Unit - I
(Random Variable)

Random variables :-

A random variable X is a function $X : S \longrightarrow R$ (S= sample space) that assigns a real number $X(s)$ to each $s \in S$ corresponding a random experiment E.

→ Range of R.V. is $-\infty$ to ∞.

→ R.V are denoted by capital letters X, Y, Z ... and their values denoted by lower case letter x, y, z ...

eg- A single fair dice is rolled and the R.V. X represents the no. that turns up

Ans- $X = \{1, 2, 3, 4, 5, 6\}$

eg- In measuring the weight of students of CS branch in a particular collage

Ans- $X = \{x \in R \mid 0 < X < \infty\}$

eg- let a coin is tossed thrice and the R.V. X denotes the no. of Tells when turn up.

Ans- $X = \{0, 1, 2, 3\}$

Discrete Random Variables ⇒

A DRV has either finite or countable infinite no. of value.

eg (i) The no. system when dice is thrown

$X = \{1, 2, 3, 4, 5, 6\}$

(ii) The no. of α-particals emitted by a radioactive source.

(iii) $X = \{x = a^2 \mid a = \text{----} -3, -2, -1, 0, 1, 2, 3 \text{---}\}$

Continuous Random Variables:—

R.V. which can take infinite no. of value in an interval is known as continuous R.V.

eg-(i) The wt. of a person.
(ii) The height of house.
(iii) $X = \{x \in R \mid 0 < x < 1\}$

PROBABILITY Distribution of Discrete Random variables

⇒

$X = x$	x_0	x_1	x_2	---
$P(x)$	P_0	P_1	P_2	---

The probability distribution of a discrete R.V. lists all the possible values that the R.V. can assume and their corresponding probability.

eg. When a rolling single fair dice then probability is

$X(x)$	1	2	3	4	5	6
P	1/6	1/6	1/6	1/6	1/6	1/6

Probability Mass function (pmf):—

Let X be a discrete random variables such that $P(X = x_i)$ then p_i is said to be pmf if its satisfies following condition:—

(i) $p_i \geq 0 \quad \forall\ i$

(ii) $\sum_i p_i = 1$

Q. Check whether the following function is pmf

(i) $P(X=x) = \dfrac{x-2}{2} \quad \forall\ x = 1, 2, 3, 4$

Ans-

$X(x)$	1	2	3	4
$P(x)$	$-1/2$	0	$1/2$	1

since $P_1 = -\frac{1}{2}$ which not satisfy the condition of pmf i.e. $p_i \geq 0 \ \forall\ i$

$P(x)$ is not pmf

Q. 4 bad oranges are mixed with 16 good oranges find the probability distribution of the no. of bad orange in the draw of 2 orange.

Ans- $X(x) = 0, 1, 2$

$$P(X=0) = \frac{{}^{16}C_2}{{}^{20}C_2} = \frac{\cancel{16}^{4} \times \cancel{15}^{3} \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{20}_{4} \times 19} = \frac{12}{19}$$

$$P(X=1) = \frac{{}^{4}C_1 \times {}^{16}C_1}{{}^{20}C_2} = \frac{\cancel{4} \times 16 \times 2 \times 1}{5\ \cancel{20} \times 19} = \frac{32}{95}$$

$$P(X=2) = \frac{{}^{4}C_2}{{}^{20}C_2} = \frac{4 \times 3 \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{20}_{5} \times 19} = \frac{3}{95}$$

$X(x)$	0	1	2
$P(x)$	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

$P(x)$ is a pmf

Distribution function $\Rightarrow$

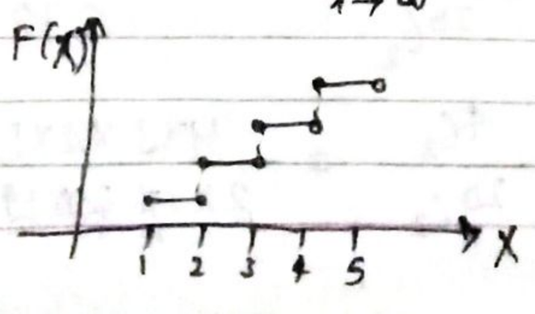
Let X be a discrete R.V. then its discrete distribution function or cumulative distribution function (cdf) is defined as

$$F(X) = P(X \leq x)$$

cdf gives the probability that the variable takes a value less than or equal to x and is defined for all real x.
If f is pmf of a discrete R.V. X with $\{x_1, x_2 ---\}$ and F is its cdf then

$$\boxed{F(X) = \sum_{x_i \leq x} f(x_i)} \qquad \boxed{F(X_{i+1}) = F(X_i) + P(x_{i+1})}$$

The D.f. $F(x)$ satisfy the following properties
(i) $F(X)$ is non-decreasing i.e. its graph never goes down. $F(a) \leq F(b)$
(ii) $0 \leq F(X) \leq 1$
(iii) $\lim_{x \to -\infty} F(X) = 0$ $\qquad \lim_{x \to \infty} F(X) = 1$



Q. Find the probability of boys & girls in families with 3 children assuming equal probability of boys & girls also give the Distribution function.

Ans —

Let R.V. $X \rightarrow$ No. of boys in the family

$X = 0, 1, 2, 3$ $\quad n = 3$

$P(\text{girls}) = \frac{1}{2}$ and $P(\text{boys}) = \frac{1}{2}$

X	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8

$$P(X = x) = {}^nC_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} = {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$$

$$P(X=0) = {}^3C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = \frac{3\times2\times1}{3\times2\times1} \times \frac{1}{8} = \frac{1}{8}$$

$$P(X=1) = {}^3C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = 3 \times \frac{1}{8} = \frac{3}{8}$$

$$P(X=2) = {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = \frac{3\times2}{1\times2\times1} \times \frac{1}{8} = \frac{3}{8}$$

$$P(X=3) = {}^3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = \frac{\lfloor 3}{\lfloor 3-3 \lfloor 3} \times \frac{1}{8} = \frac{1}{8}$$

Distribution function for x

$$F(X_{i+1}) = F(X_i) + P(x_{i+1})$$

X	0	1	2	3
$F(X)$	1/8	$1/8 + 3/8 = \frac{1}{2}$	$\frac{1}{2} + \frac{3}{8} = \frac{7}{8}$	$\frac{7}{8} + \frac{1}{8} = 1$

Probability distribution of a Continuous R.V. ⇒

A continuous R.V. can assume any value over an interval or intervals because the no. of value contain in any interval is infinite.
As the no. of events are infinitally large the probability that the perticular event will occur is practically zero.
Here instant of finding the probability at a perticular value of x we find the probability of X calling in a small interval.
Thus we defined the continuous probability distribution of X by the function $F(X)$ such that the probability of x lying in the small interval

$$\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right) \text{ is } f(x)\,dx$$

i.e. $$P\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right) = f(x)\,dx$$

PROBABILITY Density function (pdf) ⇒

The function $f(x)$ for a continuous R.V. X is said to be probability density function provided its satisfies the following condition :-

(i) $f(x) \geq 0$, $-\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

and $P(a \le X \le b) = \int_a^b f(x)\,dx$

Note:- When X is continuous R.V.

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b)$$

Q. If $f(x) = \begin{cases} x e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(i) Show that $f(x)$ is a pdf

(ii) find its distribution function $F(x)$

Ans-

(i) $\int f(x)\cdot dx = \int_0^\infty x e^{-x^2/2}\cdot dx$

$\frac{x^2}{2} = t$

$x \cdot dx = dt$

$\int_0^\infty e^{-t}\cdot dt = \cancel{\left[e^{-t/2}(t) + 2\int e^{-t/2}\cdot t\,dt\right]}$

$= -e^{-t}\cancel{(t+1)} \quad - e^{-t}$

$= -\cancel{e^{-x^2/2}}($

$= \left[-e^{-x^2/2}\right]_0^\infty = 1$

$\therefore f(x)$ is a pdf

(ii) $\because f(x) = 0$ at $x < 0$

if $x \geq 0 = \int_0^x f(x)\,dx = \int_0^x x e^{-x^2/2}\cdot dx$

$= 1 - e^{-x^2/2}$

$\therefore$ Distribution function

$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x^2/2} & , x \geq 0 \end{cases}$

Q. The diameter of an electric cable say X is assumed to be a continuous R.V. with pdf

$$f(x) = 6x(1-x) \quad , \quad 0 \le x \le 1$$

(i) Check the above given in a pdf

(ii) Determine a number 'b' such that

$$P(X<b) = P(X>b)$$

Ans- Continuous R.V $\quad X \rightarrow$ Diameter of an electric cable

$\int f(x).dx$

(i) $\quad f(x) \geq 0 \quad \forall \; 0 \le x \le 1$

$$\int_0^1 f(x).dx = \int_0^1 (6x - 6x^2)\,dx$$

$$\left[3x^2 - 2x^3\right]_0^1 = 3-2-0+0 = 1$$

$f(x) \geq 0$ & $\int f(x)dx = 1$

$\therefore$ f(x) is pdf

(ii) $$P(X<b) = \int_0^b f(x).dx = 6\int_0^b (x-x^2)\,dx$$

$$3b^2 - 2b^3$$

Now

$$P(X>b) = 1 - P(X<b)$$

$$(\because \; P(X>b) = P(X<b))$$

$$2P(X<b) = 1$$

$$P(X<b) = \frac{1}{2}$$

$$3b^2 - 2b^3 = \frac{1}{2}$$

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$$6b^2 - 4b^3 = 1$$

$$\cancel{b^2(6b - 4b) = 1}$$

$$6b^2 - 4b^3 - 1 = 0$$

$$4b^3 - 4b^2 - 2b^2 - 1 = 0$$

$$4b^3 - 4b^2 - 2b^2 - 1 + 2b - 2b + 2 - 2 = 0$$

$$4b^2(b-1) - 2b(b-1) - 2(b-1) - 3 = 0$$

$$\cancel{(b-1)(4b^2 - 2b - 2) - 3 = 0}$$

$$\left(b - \frac{1}{2}\right)(4b^2 - 4b - 2) = 0$$

$$(2b-1)(2b^2 - 2b - 1) = 0$$

$$b = \frac{1}{2}, \quad \frac{2 \pm \sqrt{4+8}}{4}$$

$$b = \frac{1}{2}, \quad \frac{1 \pm \sqrt{3}}{2}$$

$\because \frac{1+\sqrt{3}}{2} > 1$ and $\frac{1-\sqrt{3}}{2} < 0$

which not valid in $0 \le x \le 1$

$$\boxed{b = \frac{1}{2}}$$

(1) Distribution function for continuous R.V. (cdf) ⇒

Let X be a continuous R.V. with pdf $f(x)$ then the function $F_X(x)$ is called the continuous distribution function or cdf of the R.V. X

$$\boxed{F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx} \quad -\infty < x < \infty$$

Properties of continuous cdf $\Rightarrow$

(i) $0 \leq F(x) \leq 1$, $-\infty < x < \infty$

(ii) $F(x) = \int_{-\infty}^{x} f(x)dx$ and $\frac{d}{dx}F(x) = f(x) \geq 0$

$\therefore$ $F(x)$ is a non decreacing function of x

(iii) $F(-\infty) = \lim_{x \to \infty} F(x) = \int_{-\infty}^{-\infty} f(x)dx = 0$

$F(\infty) = \lim_{x \to \infty} F(x) = \int_{-\infty}^{\infty} f(x)dx = 1$

(iv) $P(a \leq x \leq b) = \int_{a}^{b} f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$

$= P(X \leq b) - P(X \leq a)$

$= F(b) - F(a)$

Q. let X be a continuous R.V. with pdf

$$f(x) = \begin{cases} ax & , 0 \leq x \leq 1 \\ a & , 1 \leq x \leq 2 \\ -ax + 3a & , 2 \leq x \leq 3 \\ 0 & , \text{elsewhere} \end{cases}$$

(i) Determine the constant a

(ii) find $P(X \leq 1.5)$

(iii) Determine the cdf and hence find $P(X \leq 2.5)$

Sol^n -

$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} -ax + 3a\,dx = 1$

$= \left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^2 + \left[-\frac{ax^2}{2} + 3ax\right]_2^3$

$$\frac{a}{2} + 2a - a \quad - \frac{9a}{2} + 9a + \frac{4a}{2} - 6a = 1$$

$$\frac{a}{2} + a \quad - \frac{9a}{2} + 5a \quad = 1$$

$$6a - 4a = 1$$

$$\boxed{a = \frac{1}{2}}$$

(ii) $P(X \leq 1.5)$

$$\int_0^1 ax \cdot dx + \int_1^{3/2} a \cdot dx$$

$$\frac{a}{2} + \frac{3}{2}a - a \quad = \quad \boxed{a = \frac{1}{2}}$$

(iii) cdf

$$P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$f(x) = 0$ at $x < 0$

if $x \geq 0 = \int_0^x f(x)\,dx = \quad = 0$

for $0 \leq x \leq 1$ $\quad F(x) = a\int_0^x x \cdot dx = \frac{ax^2}{2} = \frac{x^2}{4}$

for $1 \leq x \leq 2$ $\quad F(x) = a\int_0^1 x\,dx + a\int_1^x dx = ax - a + \frac{a}{2} = \frac{1}{4} + \frac{x-1}{2}$

~~for $(2 < x \leq 3)$ $\quad F(x) = -a\int_2^x x - 3\,dx = -a\left[\frac{x^2}{2} - 3x - 2 + 6\right]$~~

~~$F(x) = -a\left[\frac{x^2}{2} - 3x + 4\right]$~~

$$F(x) = \begin{cases} ax^2/2 & 0 \leq x \leq 1 \\ a & 1 \leq x \leq 2 \\ \cancel{-\frac{ax^2}{2} - 3ax + 8a} & 2 \leq x \leq 3 \end{cases}$$

$$-\frac{ax^2}{2} + 3ax - 4a$$

for $x \geq 3$ $\quad F(x) = \int_0^{\infty} f(x)\,dx = 1$

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for $2 \leq x \leq 3$ $\quad = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (-ax+3a)\,dx =$

$$\left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^2 + \left[\frac{-ax^2}{2} + 3ax\right]_2^x$$

$$\frac{a}{2} + a - \frac{ax^2}{2} + 3ax + 2a - 6a$$

$$\frac{3a}{2} - \frac{ax^2}{2} + 3ax - 4a = \frac{-5a}{2} - ax\left(\frac{x}{2} - 3a\right)$$

$$= \frac{-5}{4} - \frac{x^2}{4} + \frac{3x}{2}$$

If $x = 2.5$

$$f(2.5) = \frac{3a}{2} - 4a - \frac{a \times 25}{8} + \frac{3a \times 5}{2}$$

$$= 5a - \frac{25a}{8} = \frac{15a}{8} = \frac{15}{16} = 0.9375$$

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2/4, & 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{4} & 1 \leq x \leq 2 \\ -\frac{5}{4} - \frac{x^2}{4} + \frac{3x}{2} & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

$$\boxed{f(2.5) = 0.9375}$$

TWO DIMENSIONAL Random Variables ⇒

Let S be the sample space associated with the random experiment E then the function

$$f : S \longrightarrow R^2$$

Where $f(s) = (X, Y)$ $\quad s \in S$

is said to be a two dimensional random variable.

If the values of (X,Y) are finite ~~value of~~ or countably infinite then (X,Y) is called two dimension discrete random variable.
And represented as (X_i, Y_j) $\forall\ i=1,2,\dots m\dots;\ j=1,2\dots n\dots$

→ If (X,Y) can assume all value in range $\mathbb{R}^2$ in xy-plane then it is called a two dimensional continuous random variable.

Probability Mass Function of (X,Y):—

Consider two dim. discrete r.v. such that

$$P(X=x_i,\ Y=y_j) = p_{ij}$$ then

p_{ij} is known as pmf if

(i) $p_{ij} \geq 0 \quad \forall\ i,j$

(ii) $\sum\limits_j \sum\limits_i p_{ij} = 1$

the set $\{x_i, y_j, p_{ij}\}$ $i=1,2,\dots m\dots,\ j=1,2\dots n\dots$ is called the joint distribution function of (X,Y)

Q. Three balls are drawn at Random without replacement from a box containing 2 white, 3 red & 4 black balls. If X denotes the no. of white drawn and Y denotes the no. of red ball drawn then find Joint probability distribution of (X,Y).

Ans- Given:— $X = 0, 1, 2$ = No. of white balls
$Y = 0, 1, 2, 3$ = No. of red balls

$$P(X=0, Y=0) = \frac{^4C_3}{^9C_3} = \frac{4 \times 3 \times 2 \times 1}{\frac{9 \times 8 \times 7}{2}} = \frac{1}{21}$$

P(3 balls are drawn out of which none is white or red)

$$P(X=1, Y=1) = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2\times 3\times 4 \times 3\times 2\times 1}{9\times 8\times 7} = \frac{2}{7}$$

$$P(X=2, Y=1) = \frac{{}^2C_2 \times {}^3C_1}{{}^9C_3} = \frac{1\times 3 \times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{28}$$

$$P(X=2, Y=0) = \frac{{}^2C_2 \times {}^4C_1}{{}^9C_3} = \frac{1\times 4 \times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{21}$$

$$P(X=0, Y=3) = \frac{{}^3C_3}{{}^9C_3} = \frac{1\times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{84}$$

$$P(X=1, Y=2) = \frac{{}^2C_1 \times {}^3C_2}{{}^9C_3} = \frac{2\times 3 \times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{14}$$

$$P(X=0, Y=1) = \frac{{}^3C_1 \times {}^4C_2}{{}^9C_3} = \frac{3\times 4\times 3 \times 3\times 2\times 1}{9\times 8\times 7 \times 2} = \frac{3}{14}$$

$$P(X=0, Y=2) = \frac{{}^3C_2 \times {}^4C_1}{{}^9C_3} = \frac{3\times 4 \times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{7}$$

$$P(X=1, Y=0) = \frac{{}^2C_1 \times {}^4C_2}{{}^9C_3} = \frac{2\times 4\times 3 \times 3\times 2\times 1}{9\times 8\times 7 \times 2} = \frac{1}{7}$$

$P(X=2, Y=2) = P(X=2, Y=3) = P(X=1, Y=3) = 0$ {As three balls are drawn}

∴ Joint Probability distribution for (X,Y) in

X \ Y	0	1	2	3
0	1/21	3/14	1/7	1/84
1	1/7	2/7	1/14	0
2	1/21	1/28	0	0

Cumulative Distribution function (cdf) ⇒

If (X,Y) is a two dimensional discrete R.V. then $F(x,y) = P\{X \le x \text{ and } Y \le y\}$ is called the ~~cdf~~ cdf of (X,Y)

i.e. $$F(x,y) = \sum_{\substack{j \\ y_j \le y}} \sum_{\substack{i \\ x_i \le x}} p_{ij}$$

$$= P(-\infty < X \le x, \; -\infty < Y \le y)$$

☆ Marginal Probability Distribution ⇒

The pmf/probability distribution of a single variable within more than one R.V. is defined as marginal pmf/probability distribution.

* If two dimensional R.V. (x_i, y_j), $i = 1,2,\dots m,\dots$, $j = 1,2,\dots n\dots$ then

→ marginal probability function of X is defined as

$$P(X = x_i) = \sum_j p_{ij} = p_{i1} + p_{i2} + \dots + p_{in} + \dots = p_{i*}$$

and the collection of pairs $\{x_i, p_{i*}\}$, $i = 1,2\dots n\dots$ is called marginal probability distribution of X.

→ Marginal probability function of Y is defined as

$$P(Y = y_j) = \sum_i p_{ij} = p_{1j} + p_{2j} + p_{3j} + \dots + p_{nj} + \dots = p_{*j}$$

☆ and the collection of pairs $\{y_i, p_{*j}\}$, $i = 1,2\dots m\dots$ is called (m pdf) of Y.

Conditional Probability distribution ⇒

Let two dimensional discrete R.V. (X,Y) then the conditional probability function of X given $Y=y_j$ is given by

$$P\left\{\frac{X=x_i}{Y=y_j}\right\}=\frac{P\{X=x_i, Y=y_j\}}{P\{Y=y_j\}}=\frac{P_{ij}}{P_{*j}}$$

and the collection of pairs $\{x_i, P_{ij}/P_{*j}\}$, $i=1,2,---m...$ is called the <u>conditional probability distribution</u> of X given $Y=y_j$.

Let two dim. discrete R.V. (X,Y) the the conditional probability function of Y given $X=x_i$ is given by

$$P\left\{\frac{Y=y_j}{X=x_i}\right\}=\frac{P\{X=x_i, Y=y_j\}}{P\{X=x_i\}}=\frac{P_{ij}}{P_{i*}}$$

and the collection of pairs $\{x_j, p_{ij}/P_{i*}\}$, $j=1,2...n...$ is called the <u>conditional probability distribution</u> of Y given $X=x_i$

Independent Random Variables ⇒

Let (X,Y) be a two dimensional R.V. Such that $P\left\{\frac{X=x_i}{Y=y_j}\right\}=P(X=x_i)$

i.e. $$\frac{P_{ij}}{P_{*j}}=P_{i*}$$

$$\Rightarrow \boxed{P_{ij}=P_{i*}\cdot P_{*j}}$$

then X and Y are said to be independent R.V.

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Q. The joint probability mass function of (X,Y) is given by $P(x,y) = K(2x+3y)$, $x=0,1,2$; $y=1,2,3$ find

(i) K

(ii) Marginal probability distribution of function (X)

(iii) Marginal ——— of Y

(iv) Conditional distribution of X given $Y=1$

(v) Conditional distribution of Y given $X=2$

(vi) The probability distribution of $(X+Y)$

Sol-

(i) $p(x,y) = K(2x+3y)$

The joint probability distribution is

$p(x=0, y=1) = 3K$, $p(x=0, y=3) = 9K$

$p(x=0, y=2) = 6K$, $p(x=1, y=1) = 5K$

$p(x=1, y=2) = 8K$, $p(x=1, y=3) = 11K$

$p(x=2, y=1) = 7K$, $p(x=2, y=2) = 10K$

$p(x=2, y=3) = 13K$

X \ Y	1	2	3
0	3K	6K	9K
1	5K	8K	11K
2	7K	10K	13K

(i) $p(x,y)$ is a pmf function then

$\therefore \sum_j \sum_i P_{ij} = 1$

$= 3k + 6k + 9k + 5k + 8k + 11k + 7k + 10 + 13k = 72k = 1$

$\boxed{K = \frac{1}{72}}$

18
24
42
30

(ii) Marginal probability distribution of X

X	$P_{i*} = \sum_j P_{ij} = P_{i1} + P_{i2} + P_{i3}$	P_{i*}
0	$P_{0*} = 3k + 6k + 9k = 18k$	$1/4$
1	$P_{1*} = 5k + 8k + 11k = 24k$	$1/3$
2	$P_{2*} = 7k + 10k + 13k = 30k$	$\rightarrow \frac{5}{12}$
	Total	1

(iii) Marginal probability distribution of Y

Y	$P_{*j} = \sum_i P_{ij} = P_{0j} + P_{1j} + P_{2j}$
1	$P_{*1} = 3K + 5K + 7K = 15K = 5/24$
2	$P_{*2} = 6K + 8K + 10k = 24k = 1/3$
3	$P_{*3} = 9k + 11k + 13k = 33k = 33/72$
	Total = 1

(iv) Conditional distribution of X given Y=1

$$P\left\{\frac{X=i}{Y=1}\right\} = \frac{P_{i1}}{P_{*1}} = \frac{P_{i1}}{15k}$$

X	$\frac{P_{i1}}{P_{*1}} = \frac{P_{i1}}{15k}$
0	$P_{01}/P_{*1} = 3k/15k = 1/5$
1	$P_{11}/P_{*1} = 5k/15k = 1/3$
2	$P_{21}/P_{*1} = 7k/15k = 7/15$
	total = 1

(v) Conditional distribution of Y given X = 2

$$P\left\{\frac{Y=j}{X=2}\right\} = \frac{P_{2j}}{P_{2*}} = \frac{P_{2j}}{30k}$$

Y	$P_{2j}/P_{2*} = P_{2j}/30k$
1	$P_{21}/30k = 7k/30k = 7/30$
2	$P_{22}/30k = 10k/30k = 1/3$
3	$P_{23}/30k = 13k/30k = 13/30$
	total = 1

(vi) Probability Distribution of (X+Y)

Let Z = X+Y

then (X+Y) can have values 1, 2, 3, 4, 5

∴ Z can have values as

X	0	0	1	0	1	2	1	2	2
Y	1	2	1	3	2	1	3	2	3
Z	1	2		3			4		5

Probability distribution for Z = X+Y

Z	$P(X+Y) = P_{ij}$
1	$P_{01} = 3k = 3/72 = 1/24$
2	$P_{02} + P_{11} = 6k + 5k = 11k = 11/72 = 11/72$
3	$P_{03} + P_{12} + P_{21} = 9k + 8k + 7k = 24k = 24/72$
4	$P_{13} + P_{22} = 11k + 10k = 21k = 21/72$
5	$P_{23} = 13k = 13/72$
	total = 1

Probability Density function of $(X, Y) \Rightarrow$

If (X, Y) is a two dimensional continuous R.V. such that

$$P\left\{ x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \le Y \le y + \frac{dy}{2} \right\}$$

$$= f(x,y)\,dx\,dy$$

then $f(x,y)$ is called the joint pdf of (X, Y) if

(i) $f(x,y) \geq 0, \quad -\infty < x < \infty, \quad -\infty < y < \infty$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

(iii) and $P\{a \le X \le b, c \le Y \le d\} = \int_c^d \int_a^b f(x,y)\,dx\,dy$

Q. Show that $f(x,y) = xy^2 + \frac{x^2}{8}, \quad 0 \le x \le 2, \quad 0 \le y \le 1$ is joint pdf.

Sol- (i) $f(x,y) \geq 0, \quad 0 \le x \le 2, \quad 0 \le y \le 1$

$$\int_0^1 \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$

$$\int_0^1 \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2 dy$$

$$\int_0^1 2y^2 + \frac{1}{3}\,dy = \frac{2}{3} + \frac{1}{3} = 1$$

$f(x,y)$ is a joint pdf.

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Joint Distribution function ⇒

If (X,Y) is a bivariate / two dimensional continuous R.V. then the joint distribution function is

$$F(x,y) = P\{X \le x \ \& \ Y \le y\} = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

also at points of continuity of $f(x,y)$

$$\boxed{\frac{\partial^2 F}{\partial x \partial y} = f(x,y)}$$

Marginal Density ⇒

Let (X,Y) be a two dimensional continuous R.V. Then the marginal density of X is defined as :

$$X = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Let (X,Y) be a two dimensional continuous R.V. then the marginal density of Y is defined as :

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx,$$

and $$P(a \le X \le b) = P(a \le X \le b, -\infty < Y < \infty) = \int_a^b f_x(x)\,dx$$

$$P(c \le Y \le d) = \int_c^d f_Y(y)\,dy$$

Conditional Density ⇒

Let (X,Y) be a two dimensional continuous R.V. then conditional density of X given Y is

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f_Y(y)}$$

similarly the conditional density of Y given X is

$$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f_X(x)}$$

Independent Continuous R.V. $\Rightarrow$

Let (X,Y) be a bivariate R.V. then X & Y are said to be independent R.V. if

$$\boxed{f(x,y) = f_X(x) \times f_Y(y)}$$

Q. The joint pdf of a bivariate R.V. (X,Y) is given by

$$f_{XY}(x,y) = \begin{cases} d(x+y), & 0<x<3,\ 0<y<3 \\ 0, & \text{otherwise} \end{cases}$$

where d is constant.

(i) Find the value of d

(ii) Find the marginal probability density function of X & Y

(iii) Are X & Y independent?

Ans - (i) $\because$ $f(x,y)$ is a pdf

$$\int_0^3 \int_0^3 d(x+y)\, dx\, dy = 1$$

$$\int_0^3 \left[d\left(\frac{x^2}{2} + xy \right) \right]_0^3 dy = 1$$

$$d \int_0^3 \left(\frac{9}{2} + 3y \right) dy = 1$$

$$d \left[\frac{9}{2}y + \frac{3y^2}{2} \right]_0^3 = 1$$

$$d \left[\frac{27}{2} + \frac{27}{2} \right] = 1$$

$$\boxed{d = \frac{1}{27}}$$

(ii) Marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_X(x) = \int_0^3 \left(\frac{x+y}{27}\right) dy$$

$$f_X(x) = \frac{1}{27}\left[xy + \frac{y^2}{2}\right]_0^3 = \frac{6x+9}{2} \times \frac{1}{27}$$

$$f_X(x) = \frac{2x}{18} + \frac{1}{6} = \frac{x}{9} + \frac{1}{6}$$

$$\cancel{P(0 \le x \le 3)} = \cancel{\int_0^3 f_X(x)\,dx} = \cancel{\int_0^3 \left(\frac{2x}{18} + \frac{1}{6}\right) dx}$$

$$\cancel{\left[\frac{x^2}{36} + \frac{x}{6}\right]_0^3} = \cancel{\frac{1}{4} + \frac{1}{2} = \frac{3}{4}}$$

Marginal pdf of Y is

$$f_Y(x) = \frac{y}{9} + \frac{1}{6}$$

(iii) $$f(x,y) = f_X(x) \times f_Y(y)$$

$$f(x,y) = \frac{1}{27}\left(3x + \frac{9}{2}\right) \times \frac{1}{27}\left(3y + \frac{9}{2}\right)$$

$$\frac{1}{27}(x+y) \neq \frac{1}{(27)^2}\left(3x + \frac{9}{2}\right)\left(3y + \frac{9}{2}\right)$$

Q. The joint pdf of a bivarite R.V. (X,Y) is

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

find

(i) Marginal density of X & Y

(ii) Conditional density of x given $Y=y$ and use it to evaluate $P\left\{\frac{X \le 1/2}{Y = 1/2}\right\}$

Ans-(i) Marginal density of X

$$f_X(x) = \int_0^1 \frac{2}{3}(x+2y)\,dy$$

$$f_X(x) = \frac{2}{3}\left[xy + y^2\right]_0^1$$

$$f_X(x) = \frac{2}{3}(x+1)$$

Marginal density of Y

$$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx$$

$$f_Y(y) = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1$$

$$= \frac{2}{3}\left[\frac{1}{2} + 2y\right]$$

$$= \frac{1}{3} + \frac{4y}{3}$$

(ii) Conditional density $\quad P\left\{\frac{X \le 1/2}{Y \le 1/2}\right\}$

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f_Y(y)}$$

$$f\left(\frac{x}{y}\right) = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(4y+1)} = \frac{2x+4y}{4y+1}$$

$$P\left\{\frac{x \le 1/2}{Y = 1/2}\right\} = \cancel{\int_0^{1/2}(2x+4y)} \quad \int_0^{1/2} f\left(\frac{x}{y=1/2}\right)dx$$

$$\int_0^{1/2} \frac{2(x+1)}{3}\,dx = \frac{2}{3}\left[\frac{x^2}{2} + x\right]_0^{1/2}$$

$$\frac{2}{3}\left[\frac{1}{8} + \frac{1}{2}\right] = \frac{2}{3}\left[\frac{1+4}{8}\right] = \frac{5}{12}$$

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Expectation ⇒

(a) Mathematical Expectation (For univariate variable) ⇒

The expectation of a R.V. X is defined as

$$\bar{X} = E(X) = \begin{cases} \sum_i x_i p_i & \text{, if } X \text{ is discrete R.V. with pmf } p_i \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{, if } X \text{ is continuous R.V. with pdf } f(x) \end{cases}$$

* If X is a R.V. and g(x) is any function of X then

$$E[g(x)] = \begin{cases} \sum_i g(x_i) p_i & \text{, if } X \text{ is discrete R.V. with pmf } p_i \\ \int_{-\infty}^{\infty} g(x_i) f(x)\,dx & \text{, if } X \text{ is continuous R.V. with pdf } f(x) \end{cases}$$

(b) Mathematical expectation (For bivariate variable) ⇒

$$E[h(x,y)] = \begin{cases} \sum_i \sum_j h(x_i, y_j) p_{ij} & \text{, for discrete R.V. with joint pmf } p_{ij} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x_i, y_j) f(x,y)\,dx\,dy & \text{, for continuous R.V. with joint pdf } f(x,y) \end{cases}$$

Properties of Mathematical Expectation :—

* $E(X_1 + X_2) = E(X_1) + E(X_2)$
* $E(X_1 X_2) = E(X_1)\,E(X_2)$

Q. Find the expectation of the number on a dice when thrown.

Sol^n Let R.V.

$X \rightarrow$ no. on dice when thrown then probability distribution is

X	1	2	3	4	5	6
P_i	1/6	1/6	1/6	1/6	1/6	1/6

$$\bar{X} = E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$E(X) = \bar{X} = 3.5$ Ay

★★★

Moments $\Rightarrow$ Moment is the measure of a course with respect to its tenden to provide rotation and the strength of ~~a~~ the tendency depends on the amount of force and distance from the ~~original~~ origin of the point at which the force is exerted.

The moments are used to describe the various characteristic of a frequency distribution like variation, central tendency, skewness, kurtosis.

The moment is defined as

$$\mu = \frac{\sum f_i x_i}{\sum f_i} \quad ; \quad i = 1, 2, --- n$$

where $f_1, f_2, --- f_n \rightarrow$ no. of forces

$x_1, x_2, --- x_n \rightarrow$ distance from the point at which force exerted

* If $x_1, x_2 --- x_n$ are n values assumed by the variate x the r^{th} moment is defined as

$$\overline{X}^r = \frac{x_1^r + x_2^r + x_3^r + \cdots + x_n^r}{n} = \frac{\sum_i x_i^r}{n} \quad \text{at } r=1$$

$$\text{first moment} = \text{mean } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

(i) Moment about the origin:—

r^{th} moment about the origin

$$\mu_r' = E[(x-0)^r] = E[x^r] = \sum_i x_i^r p_i$$

or

$$\mu_r' = \frac{\sum f_i x^r}{\sum f_i} \qquad r = 0, 1, 2, 3 ---$$

Case - I At $r=0$

$$\mu_0' = \sum_i p_i = 1 \Rightarrow \boxed{\mu_0' = 1}$$

Case - II At $r=1$

$$\mu_1' = \sum x_i p_i = E(x) = \overline{x} \Rightarrow \boxed{\mu_1' = \overline{x}}$$

Case - III At $r=2$

$$\mu_2' = \sum x_i^2 p_i = E(x^2) \Rightarrow \boxed{\mu_2' = E(x^2)}$$

Case - IV At $r=3$

$$\mu_3' = \sum x_i^3 p_1 = E(x^3) \Rightarrow \boxed{\mu_3' = E(x^3)}$$

Case - V At $r=4$

$$\mu_4' = \sum x_i^4 p_i = E(x^4) \Rightarrow \boxed{\mu_4' = E(x^4)}$$

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(b) Central Moments / Moment about the mean ⇒

* The r^{th} moment about the mean $\bar{x}$ is

$$\mu_r = E[(x-\bar{x})^r] = \sum_i (x_i - \bar{x})^r p_i \quad \text{(for discrete R.V.)}$$

$$\mu_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f(x)\,dx \quad \text{(for continuous)}$$

or $$\mu_r = \frac{\sum_i f_i (x_i - \bar{x})^r}{\sum_i f_i}, \quad r = 0, 1, 2 \ldots$$

Case I At $r = 0$

$$\mu_0 = \sum_i p_i = 1 \Rightarrow \boxed{\mu_0 = 1}$$

Case II At $r = 1$

$$\mu_1 = \sum_i (x_i - \bar{x}) p_i$$

$$= \sum_i x_i p_i - \sum_i \bar{x} p_i$$

$$\mu_1 = \bar{x} - \bar{x}(1) \Rightarrow \boxed{\mu_1 = 0}$$

Case III At $r = 2$

$$\mu_2 = \sum_i (x_i - \bar{x})^2 p_i$$

$$\mu_2 = \sum_i (x_i^2 - 2x_i\bar{x} + \bar{x}^2) p_i$$

$$\mu_2 = \sum_i x_i^2 p_i - 2\bar{x}\sum_i x_i p_i + \bar{x}^2 \sum_i p_i$$

$$\mu_2 = E(x^2) - 2\bar{x}^2 + \bar{x}^2$$

$$\mu_2 = E(x^2) - \bar{x}^2$$

$$\mu_2 = E(x^2) - (E(x))^2$$

$$\sigma^2 = \boxed{\mu_2 = \mu_2' - (\mu_1')^2}$$

Case IV At $r = 3$

$$\mu_3 = \sum_i (x_i - \bar{x})^3 p_i$$

$$\mu_3 = \sum_i (x_i^3 - 3x_i^2 \bar{x} + 3x_i \bar{x}^2 - \bar{x}^3) p_i$$

$$\mu_3 = \sum_i x_i^3 p_i - 3\bar{x} \sum_i x_i^2 p_i + 3\bar{x}^2 \sum_i x_i p_i - \bar{x}^3 \sum_i p_i$$

$$\mu_3 = E(x^3) - 3\bar{x} E(x^2) + 3\bar{x}^2 \bar{x} - \bar{x}^3$$

$$\mu_3 = E(x^3) - 3\bar{x} E(x^2) + 2\bar{x}^3$$

$$\mu_3 = E(x^3) - 3E(x) E(x^2) + 2(E(x))^3$$

$$\boxed{\mu_3 = \mu_3' - 3\mu_1' \mu_2' + 2\mu_1'^3}$$

Case V At $r = 4$

$$\mu_4 = \sum_i (x_i - \bar{x})^4 p_i$$

$$\mu_4 = \sum_i (x_i^2 - 2x_i\bar{x} + \bar{x}^2)(x_i^2 - 2x_i\bar{x} + \bar{x}^2) p_i$$

$$\boxed{\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}$$

(c) Moment about any point $x = A \Rightarrow$

* The r^{th} moment ~~about the~~ about any point $x = A$

$$\boxed{\mu_r'' = E[(x-A)^r] = \sum_i (x_i - A)^r p_i} \rightarrow \text{(For discrete R.V.)}$$

$$\boxed{\mu_r'' = \int_{-\infty}^{\infty} (x-A)^r f(x)\,dx} \rightarrow \text{(for continuous R.V.)}$$

or $$\mu_r'' = \frac{\sum_i f_i (x_i - A)^r}{\sum_i f_i}, \quad r = 0, 1, 2 \ldots$$

Case-I At $r = 0$ $\quad \mu_0'' = \sum_i p_i = 1 \Rightarrow \boxed{\mu_0'' = 1}$

Case-II

At $r=1$

$$\mu_1'' = \sum_i (x_i - A)^1 p_i = \sum_i x_i p_i - \sum_i A p_i = \bar{x} - A$$

$$\mu_1'' = \bar{x} - A$$

$$\boxed{\mu_1'' = E(x) - A}$$

Case-III

At $r=2$

$$\mu_2'' = \sum_i (x_i - A)^2 p_i = \sum_i (x_i^2 - 2x_i A + A^2) p_i$$

$$\mu_2'' = \mu_2 + \mu_1''^2$$

$$\mu_2'' = E(x^2) - 2A\,E(x) + A^2$$

$$\mu_2'' = \mu_2 + \cancel{A(E(x)-A)} + A(A - 2E(x))$$

$$\mu_2 = \mu_2 - A(\mu_1'')$$

$$\boxed{\mu_2'' = \mu_2 + \mu_1''^2}$$

Case IV

At $r=3$

$$\boxed{\mu_3'' = \mu_3 + 3\mu_2 \mu_1'' + \mu_1''^3}$$

Case V

At $r=4$

$$\boxed{\mu_4'' = \mu_4 + 4\mu_3 \mu_1'' + 6\mu_2 \mu_1''^2 + \mu_1''^4}$$

Q. The first four moment of a distribution about the value 5 are -4, 22, -117 and 560, obtain the moment about (i) Mean and (ii) origin

Sol^n:- Given $A = 5$

$$\mu_1'' = -4, \quad \mu_2'' = 22, \quad \mu_3'' = -117, \quad \mu_4'' = 560$$

(i) Mean :—

$$\mu_1'' = \bar{x} - A \Rightarrow \bar{x} = \mu_1'' + A$$

$$\bar{x} = -4 + 5 = 1 \Rightarrow \bar{x} = 1$$

Moments about the mean :—

$$\boxed{\mu_1 = 0}$$

$$\mu_2'' = \mu_2 + \mu_1''^2$$

$$\mu_2 = \mu_2'' - \mu_1''^2 = 22 - 16 = \cancel{38}\ 6$$

$$\boxed{\cancel{\mu_2 = 38}} \qquad \boxed{\mu_2 = 6}$$

$$\cancel{\mu_3'' = \mu_3 + 3\mu_2\mu_1'' - 2\mu_1''^3}$$

$$\cancel{\mu_3'' - 3\mu_2\mu_1'' + 2\mu_1''^3 = \mu_3}$$

$$\cancel{\mu_3 = -117 - 3\times 6\times(-4) + 2(-4)^3}$$

$$\cancel{\mu_3 = -117 + 72 - 128}$$

$$\cancel{-245 + 264 = -173}$$

$$\mu_3'' = \mu_3 + 3\mu_2\mu_1'' + \mu_1''^3$$

$$\mu_3 = \mu_3'' - 3\mu_2\mu_1'' - \mu_1''^3$$

$$\mu_3 = -117 - 3(6)(-4) - (-4)^3$$

$$\mu_3 = -117 + 72 + 64$$

$$\mu_3 = 136 - 117 = 19 \qquad \boxed{\mu_3 = 19}$$

$$\mu_4'' = \mu_4 + 4\mu_3\mu_1'' + 6\mu_2\mu_1''^2 + \mu_1''^4$$

$$\mu_4 = \mu_4'' - 4\mu_3\mu_1'' - 6\mu_2\mu_1''^2 - \mu_1''^4$$

$$\mu_4 = 560 - 4(19)(-4) - 6(6)(-4)^2 - (-4)^4$$

$$\mu_4 = 560 + 304 - 576 - 256$$

$$\boxed{\mu_4 = 32}$$

Moments about the origin:—

$$\mu_1' = \bar{x} = 1 \Rightarrow \boxed{\mu_1' = 1}$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \qquad \mu_2' = \mu_2 + \mu_1'^2$$

$$\mu_2' = 6 + 1 = \boxed{\mu_2' = 7}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3$$

$$\mu_3' = \mu_3 + 3\mu_1'\mu_2' - 2\mu_1'^3$$

$$\mu_3' = 19 + 3(1)(7) - 2(1)^3$$

$$\mu_3' = 19 + 21 - 2 = 38$$

$$\boxed{\mu_3' = 38}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$\mu_4' = \mu_4 + 4\mu_3'\mu_1' - 6\mu_2'\mu_1'^2 + 3\mu_1'^4$$

$$\mu_4' = 32 + 4(38)(1) - 6(7)(1)^2 + 3(1)^4$$

$$\mu_4' = 32 + 152 - 42 + 3$$

$$\mu_4' = 187 - 42$$

$$\boxed{\mu_4' = 145}$$

Q. Calculate the first four moments about $x=5$ and about the mean of the following distribution also calculate β_1 and β_2.

x	1	2	3	4	5	6	7	8	9
y	1	6	13	25	30	22	9	5	2

Ans –

x	y	$x-5$	$y(x-5)$	$y(x-5)^2$	$y(x-5)^3$	$y(x-5)^4$
1	1	−4	−4	16	−64	256
2	6	−3	−18	54	−162	486
3	13	−2	−26	52	−104	208
4	25	−1	−25	25	−25	25
5	30	0	0	0	0	0
6	22	1	22	22	22	22
7	9	2	18	36	72	144
8	5	3	15	45	135	405
9	2	4	8	32	128	512
	$\Sigma y = 113$		$\Sigma y(x-5) = -10$	$\Sigma y(x-5)^2 = 282$	$\Sigma y(x-5)^3 = 2$	$\Sigma y(x-5)^4 = 2058$

$$\mu_1'' = \frac{\Sigma y(x-5)}{\Sigma y} = \frac{-10}{113} = -0.088$$

$$\mu_2'' = \frac{\Sigma y(x-5)^2}{\Sigma y} = \frac{282}{113} = 2.496$$

$$\mu_3'' = \frac{\Sigma y(x-5)^3}{\Sigma y} = \frac{2}{113} = 0.018$$

$$\mu_4'' = \frac{\Sigma y(x-5)^4}{\Sigma y} = \frac{2058}{113} = 18.212$$

Now moments about the mean $\bar{x}$

$\mu_1 = 0$, $\quad \mu_2 = \mu_2'' - \mu_1''^2 = 2.496 - (0.088)^2 = 2.488$

$$\mu_3 = \mu_3'' - 3\mu_2\mu_1'' - \mu_1''^3$$

$$\mu_3 = 0.018 - 3(2.488)(-0.088) - (-0.088)^3$$

$$\mu_3 = 0.018 + 0.6568 + 0.00068$$

$$\boxed{\mu_3 = 0.675\cancel{58}8}$$

$$\mu_4 = \mu_4'' - 4\mu_3\mu_1'' - 6\mu_2\mu_1''^2 - \mu_1''^4$$

$$\mu_4 = 18.212 - 4(0.6755)(-0.088) - 6(2.488)(-0.088)^2 - (0.088)^4$$

$$\mu_4 = 18.212 + 0.2377 - 0.1156 - 0.5984 \times 10^{-4}$$

$$\mu_4 = 18.3341$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0.029 \simeq 0.03$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 2.928 \simeq 3$$

Moment Generating function (mgf):- MGF helps us to find moment without evaluating integral or sum)

The moment generating function of a random variable X having the probability function $f(x)$ is given by $M_x(t)$

$$M_x(t) = E[e^{tx}] = \begin{cases} \sum e^{tx} f(x), & \text{for discrete R.V.} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{for continuous R.V.} \end{cases}$$

∴ MGF about the origin :—

Now

$$M_x(t) = E[e^{tx}] = E\left[1 + tx + \frac{t^2x^2}{\lfloor 2} + \cdots + \frac{t^r x^r}{\lfloor r} + \cdots\right]$$

$$= 1 + tE[x] + \frac{t^2}{\lfloor 2} E(x^2) + \cdots \frac{t^r E(x^r)}{\lfloor r} + \cdots$$

$$\boxed{M_x(t) = 1 + t\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \cdots + \frac{t^r}{\lfloor r}\mu_r' + \cdots}$$

$$\boxed{\mu_r' = \left[\frac{d^r}{dt^r}(M_x(t)) \text{ about the origin}\right]_{t=0}}$$

Similarly :—

mgf about the mean $\bar{x}$ which generates central moment

$$M_x(t) = E[e^{t(x-\bar{x})}] = E\left[1 + t(x-\bar{x}) + \frac{t^2(x-\bar{x})^2}{\lfloor 2} + \cdots \right.$$

$$\left. \cdots + \frac{t^r(x-\bar{x})^r}{\lfloor r} + \cdots\right]$$

$$= 1 + tE(x-\bar{x}) + \frac{t^2}{\lfloor 2}E(x-\bar{x})^2 + \cdots + \frac{t^r E(x-\bar{x})^r}{\lfloor r} + \cdots$$

$$\boxed{M_x(t)\,(\text{about } \bar{x}) = 1 + t\mu_1 + \frac{t^2}{\lfloor 2}\mu_2 + \cdots \frac{t^r}{\lfloor r}\mu_r + \cdots}$$

and $$\boxed{\mu_r = \left[\frac{d^r}{dt^r} M_x(t) \text{ about } \bar{x}\right]_{t=0}}$$

Similarly, mgf about any point A which generates central moment

$$M_x(t) = E\left[e^{t(x-A)}\right] = E\left[1 + t(x-A) + \frac{t^2}{\lfloor 2}(x-A)^2 + \cdots + \frac{t^r}{\lfloor r}(x-A)^r + \cdots\right]$$

$$= 1 + tE(x-A) + \frac{t^2}{\lfloor 2}E((x-A)^2) + \cdots \frac{t^r}{\lfloor r}E(x-A)^r + \cdots$$

$$\boxed{M_x(t)\left(\text{about point } A\right) = 1 + t\mu_1'' + \frac{t^2}{\lfloor 2}\mu_2'' + \cdots \frac{t^r}{\lfloor r}\mu_r'' + \cdots}$$

and $$\boxed{\mu_r'' = \left[\frac{d^r}{dt^r} M_x(t) \text{ about point } A\right]_{t=0}}$$

→ Moment generating function suffers from some drawback as though all moments are present but mgf doesn't generate all of them. and hence has restricted use in ~~strati~~ statistics.

→ The mgf of a distribution if exists, uniquely determines the distribution.

$M_x(t) = M_y(t) \Leftrightarrow$ X & Y are identically distribution.

Date: / /

Q. Let the R.V. x assume the value 'r' with the probability law
$$P(X=r) = q^{r-1}p \quad , r=1,2,3 \ldots$$
find the mgf of x and hence its mean & variance, verify it by finding the mean from usual definition.

Ans - $P(X=r) = q^{r-1}p \quad , r=1,2,3 \ldots$

Moment generation function for R.V. X
$$M_X(t) = E(e^{tX}) = \sum_{r=1}^{\infty} e^{tr} p_r$$
$$= \sum_{r=1}^{\infty} e^{tr} q^{r-1} p$$
$$= \frac{p}{q} \sum_{r=1}^{\infty} e^{tr} q^r$$
$$= \frac{p}{q}\left[e^t q + e^{2t} q^2 + \ldots \right]$$
$$= \frac{p}{\cancel{q}} \cancel{q} e^t \left(1 + e^t q + e^{2t} q^2 + \ldots \right)$$
$$= \frac{pe^t}{1-e^t q}$$
$$\boxed{M_X(t) = \frac{pe^t}{1-e^t q}}$$

Mean $\bar{x} = \mu_1' = \frac{d}{dt}\left[(M_X(t))\right]_{t=0}$
$$\left[\frac{d}{dt}\left(\frac{pe^t}{1-e^t q}\right)\right]_{t=0}$$
$$= \left[\frac{(1-e^t q)\, pe^t - pe^t(-e^t q)}{(1-e^t q)^2}\right]_{t=0}$$
$$= \left[\frac{pe^t(1-\cancel{e^t q}+\cancel{e^t q})}{(1-e^t q)^2}\right]_{t=0}$$

$$\left[\because \sigma^2 = E(x^2) - [E(x)]^2\right]$$

$$\cancel{M_x(t)}\ \bar{x} = \mu_1' = \frac{p}{(1-q)^2} \quad (\because 1-q=p)$$

$$\bar{x} = \boxed{\mu_1' = \frac{p}{p^2} = \frac{1}{p}}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}\left(\frac{pe^t}{(1-e^tq)^2}\right)\right]_{t=0}$$

$$\mu_2' = \left[\frac{(1-e^tq)^2 pe^t - pe^t(2)(1-e^tq)(-e^tq)}{(1-e^tq)^4}\right]_{t=0}$$

$$\mu_2' = \frac{pe^t\left[1-2e^tq + (e^tq)^2 + 2e^tq - 2(e^tq)^2\right]}{(1-e^tq)^4}$$

$$\mu_2' = \left[\frac{pe^t(1+e^tq)}{(1-e^tq)^3}\right]_{t=0}$$

$$\mu_2' = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

$$\boxed{\mu_2' = \frac{1+q}{p^2}}$$

$$\because \text{Variance}(\sigma^2) = \mu_2' - \mu_1'^2 = E(x^2) - [E(x)]^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Now Mean

$$\bar{x} = E(x) = \sum x p_r = \sum_{r=1}^{\infty} r q^{r-1} p$$

$$\bar{x} = \frac{p}{q}\sum_{r=1}^{\infty} r q^{r} = \frac{p}{q}\left[q + 2q^2 + 3q^3 + \cdots\right]$$

$$\bar{x} = p(1 + 2q + 3q^2 + \cdots)$$

$$\bar{x} = p(1-q)^{-2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{\bar{x} = \frac{1}{p}}$$

Kerl Peason's Coefficient ⇒

(i) β - coefficient :-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2}$$

(ii) γ coefficient :-

$$\gamma_1 = \pm\sqrt{\beta_1}, \quad \gamma_2 = \beta_2 - 3$$

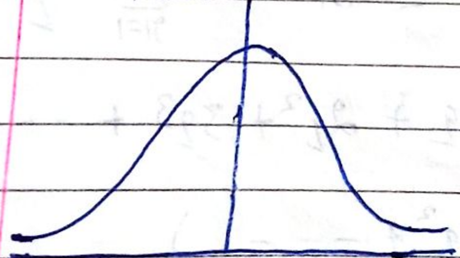
Skewness :-

→ Skewness is the definition of symmetric

→ Skewness means lack of symmetric. A distribution is said to be skewed if

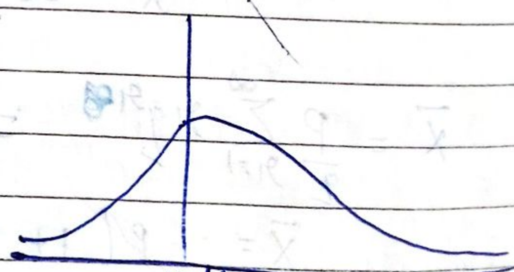
Mean ≠ Median ≠ Mode

→ A distribution is said to be skewed if the curve drawn with the help of given data is not symmetrical but stressed more to one side the other.



Mean = Median = Mode

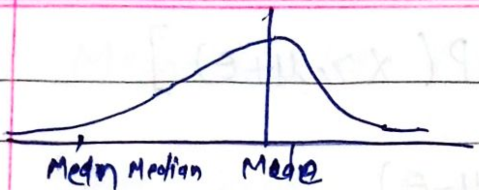
Symmetrical Distribution



Mode Median Mean

Positively Skewed Distribution

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\, dx$$



Negatively Skewed Distribution

* If $\beta_1 = 0$, then the distribution must be symmetric

~~KURTOSIS~~ ⇒

CHEBYSHEV'S Inequality ⇒ If X is a R.V. with $E(X) = \mu$ and $Var(X) = \sigma^2$

then

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \text{where } \epsilon > 0$$

Proof : Let X be continuous R.V with pdf $f(x)$

then $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$

$$\sigma^2 = \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f(x) dx + \int_{\mu-\epsilon}^{\mu+\epsilon} (x-\mu)^2 f(x) dx + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f(x) dx$$

$$\geq \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f(x) dx + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f(x) dx \quad \text{—} ①$$

In I integral

$x \leq \mu - \epsilon \Rightarrow (x-\mu)^2 \geq \epsilon^2$

In II integral

$x \geq \mu + \epsilon \Rightarrow (x-\mu)^2 \geq \epsilon^2$

By eq^n ①

$$\sigma^2 \geq \epsilon^2 \left[\int_{-\infty}^{\mu-\epsilon} f(x) dx + \int_{\mu+\epsilon}^{\infty} f(x) dx \right]$$

$$\Rightarrow \sigma^2 \geq \epsilon^2 \left[P(X \leq \mu - \epsilon) + P(X \geq \mu + \epsilon) \right]$$

$$\Rightarrow \sigma^2 \geq \epsilon^2 \; P(\mu + \epsilon \leq X \leq \mu - \epsilon)$$

$$\Rightarrow \sigma^2 \geq \epsilon^2 \; P(\epsilon \leq X - \mu \leq -\epsilon)$$

$$\Rightarrow \sigma^2 \geq \epsilon^2 \; P(|X - \mu| \geq \epsilon)$$

$$\Rightarrow \frac{\sigma^2}{\epsilon^2} \geq P(|X - \mu| \geq \epsilon)$$

$$\Rightarrow \boxed{P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}}$$

$$\Rightarrow \boxed{P(|X - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2}} \quad , \quad \epsilon > 0$$

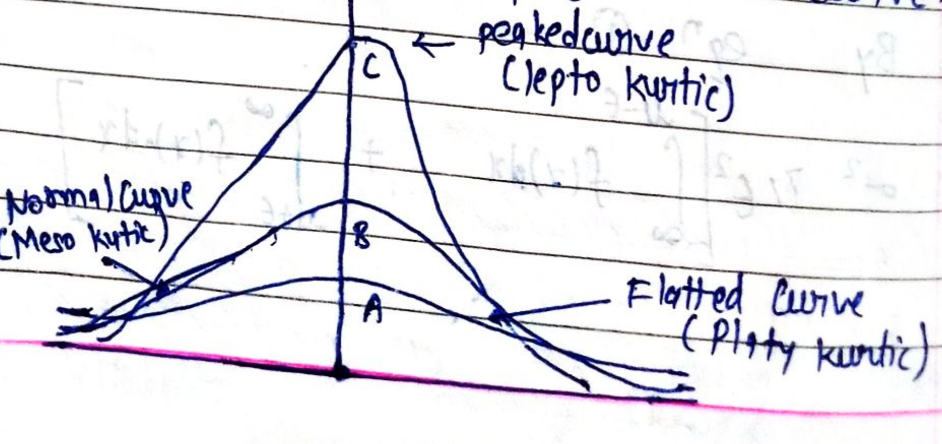
If $\mu = 0$

$$\Rightarrow \boxed{P(|X| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}}$$

or

$$\boxed{P(|X| < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2}}$$

KURTOSIS ⇒ Kurtosis is the flatness or convexi of the curve. It enables us to determine whether the curve is flat or peaked curve. There are 3 possiblity of the curve.



(i) Meso kurtic Curve :— The normal curve B is known as meso kurtic curve.

(ii) Lepto kurtic Curve :— The curve C is more peaked than the curve B then it will be lepto kurtic curve.

(iii) Platy kurtic Curve :— The curve A is more flat than the curve B then it will be platy kurtic curve.

* Coefficient β_2 can be measured the kurtosis, we have $\gamma_2 = \beta_2 - 3$

Case-1 If $\beta_2 = 3$ then $\gamma_2 = 0$ then the curve will be meso kurtic.

Case 2 If $\beta_2 > 3$ then $\gamma_2 > 0$ then the curve will be leptokurtic

Case 3 If $\beta_2 < 3$ then $\gamma_2 < 0$ then the curve will be platy kurtic

Q. Calculate the first 4 moment about the mean for the following distribution and also find β_1 and β_2

x	0	1	2	3	4	5	6	7	8
y	1	8	28	56	70	56	28	8	1

Ay -

$\mu_1 = 0$

$$\mu_2 = \frac{\sum y(x-\bar{x})^2}{\sum y}$$

$$\mu_3 = \frac{\sum y(x-\bar{x})^3}{\sum y}$$

$$\mu_4 = \frac{\sum y(x-\bar{x})^4}{\sum y}$$

$$\bar{x} = \frac{\Sigma xy}{\Sigma y} = \frac{0+8+56+168+280+280+56+8+176}{1+8+28+56+70+56+28+8+1}$$

$$\bar{x} = \frac{1024}{256} = 4$$

Imp

x	y	xy	$x-\bar{x}$	$y(x-\bar{x})^2$	$y(x-\bar{x})^3$	$y(x-\bar{x})^4$
0	1	0	-4	16	-64	256
1	8	8	-3	72	-216	648
2	28	56	-2	112	-224	448
3	56	168	-1	56	-56	56
4	70	280	0	0	0	0
5	56	280	1	56	56	56
6	28	176	2	112	224	448
7	8	56	3	72	216	648
8	1	8	4	16	64	256
	$\Sigma y = 256$	$\Sigma xy = 1024$	$\Sigma x-\bar{x} = 0$	$\Sigma y(x-\bar{x})^2 = 512$	$\Sigma y(x-\bar{x})^3 = 0$	$\Sigma y(x-\bar{x})^4 = 2816$

$\mu_1 = 0$

$$\mu_2 = \frac{512}{256} = 2$$

$\mu_3 = 0$

$$\mu_4 = \frac{2816}{256} = 11$$

$$\beta_1 = \cancel{\frac{\mu_2^3}{\mu_2^3}} \quad \frac{\mu_3^2}{\mu_2^3} = \frac{(0)^2}{(2)^3} = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2.75$$

As $\beta_1 = 0$, then ~~distribution must be~~ curve is symmetric about the mean & ~~symmetric~~ $\beta_2 < 3$ hence curve is planty kurtic in nature.

Unit - 2

Binomial Distribution ⇒

Let an experiment we repeated n times each of its trial are independent and each have two outcomes either success(s) or failure (F).

$p \to$ probability of Success

$q \to$ probability of failure

$$\boxed{q = 1-p}$$

A R.V. X associated with this experiment is said to follow binomial distribution if its pmf is

$$\boxed{P(X=r) = P(r) = {}^nC_r \, p^r \, q^{n-r}} \qquad r = 0,1,2 \dots n$$

$n, p \to$ parameter of distribution

$X \sim B(n,p) \to$ Binomial Variate

To prove that above $P(X=r)$ is a pmf

(i) $P(X=r) \geq 0 \quad \forall \; r = 0,1,2 \dots n, \; p,q \geq 0$

(ii) $\sum_{r=0}^{n} P(X=r) = \sum_{r=0}^{n} {}^nC_r \, p^r q^{n-r}$

$= {}^nC_0 q^n + {}^nC_1 p q^{n-1} + {}^nC_2 p^2 q^{n-2} + \dots + {}^nC_n p^n q^0$

$= (p+q)^n = (1)^n = 1$

$\Rightarrow$ $P(X=r)$ is a pmf

* If n trials constitute a set and consider N such ~~##~~ sets then the no. of sets in which we get exactly r success

'r' success $= N \times {}^nC_r p^r q^{n-r}$ $\quad r = 0, 1, 2 \dots n$

Mean $\mu = E(x) = \sum_{r=0}^{n} r \cdot p(x=r)$

variance $\sigma^2 = E(x^2) - [E(x)]^2$

Mean & Variance of Binomial Distribution ⇒

$$\text{Mean } (\mu) = E(x) = \sum_{r=0}^{n} r \cdot p(r)$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{\lfloor n}{\lfloor n-r \lfloor r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \cancel{r} \cdot \frac{\lfloor n}{\cancel{r} \lfloor r-1 \lfloor n-r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{\lfloor n}{\lfloor r-1 \lfloor n-r} p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{n \lfloor n-1}{\lfloor r-1 \lfloor (n-1)-(r-1)} p^{(r-1)+1} q^{(n-1)-(r-1)}$$

$$= np \sum_{r=1}^{n} {}^{n-1}C_{r-1} p^{r-1} q^{(n-1)-(r-1)}$$

$$= np \left[{}^{n-1}C_0 p^0 q^{n-1} + {}^{n-2}C_1 p^1 q^n + \dots {}^{n-1}C_n p^{n-1} q^0 \right]$$

$$= np (p+q)^{n-1} \quad (\because p+q = 1)$$

$$\boxed{\text{Mean } (\mu) = np}$$

$$E(x^2) = \sum_{r=0}^{n} r^2 \cdot P(r)$$

$$E(x^2) = \sum_{r=0}^{n} [r(r-1)+r] \; P(r)$$

$$E(x^2) = \sum_{r=0}^{n} [r(r-1)+r] \; {}^nC_r \, p^r q^{n-r}$$

$$E(x^2) = \sum_{r=0}^{n} [r(r-1)+r] \frac{\lfloor n}{\lfloor n-r \, \lfloor r} p^r q^{n-r}$$

$$E(x^2) = \sum_{r=0}^{n} [r(r-1)+r] \frac{n(n-1) \lfloor n-2}{r(r-1) \lfloor r-2 \, \lfloor (n-2)-(r-2)} p^{(r-2)+2} q^{(n-2)-(r-2)}$$

$$E(x^2) = \sum_{r=0}^{n} \frac{n(n-1) \lfloor n-2}{\lfloor r-2 \, \lfloor (n-2)-(r-2)} p^{(r-2)+2} q^{(n-2)-(r-2)} + \sum_{r=0}^{n} \frac{n \lfloor n-1}{\lfloor r-1 \, \lfloor (n-1)-(r-1)} p^{(r-1)+1} q^{(n-1)-(r-1)}$$

$$E(x^2) = n(n-1)p^2 \sum_{r=2}^{n} \frac{\lfloor n-2}{\lfloor r-2 \, \lfloor (n-2)-(r-2)} p^{r-2} q^{(n-2)-(r-2)} + np \sum_{r=1} \frac{\lfloor n-1}{\lfloor r-1 \, \lfloor (n-1)-(r-1)} p^{r-1} q^{(n-1)-(r-1)}$$

$$E(x^2) = n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2} \, p^{r-2} q^{(n-2)-(r-2)} + np \sum_{r=1}^{n} {}^{n-1}C_{r-1} \, p^{r-1} q^{(n-1)-(r-1)}$$

$$E(x^2) = n(n-1)p^2 \, (p+q)^{n-2} + np \, (p+q)^{n-1}$$

$$E(x^2) = n^2p^2 - np^2 + np$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$\sigma^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\sigma^2 = np(1-p)$$

$$\boxed{\sigma^2 = npq}$$

Standard deviation $\sigma = \sqrt{npq}$

mgf :- $[1-p+pe^t]^n$

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Fitting of Binomial Distribution :—
Recurrence formula for the probabilities of Binomial distribution

$$\boxed{P(r+1) = \frac{(n-r)\,p}{(r+1)\,q}\,P(r)}$$

For a given frequency distribution x_i/f_i, $r=1,2,\ldots$ and $\sum f_i = N$

* Mean of Binomial distribution $np = \dfrac{\sum x_i f_i}{\sum f_i}$

* Binomial frequency distribution from successive terms
$= N \times P(r)$

Q.1 If 10% of the pens manufactured by the comp are defective find the probability that a box c 12 pens contain
(i) Exactly 2 defective pens.
(ii) Atleast two defective pens
(iii) No defective pens.

A - Let R.V.
$X =$ no. of defective pens
$n = 12$, $p = \frac{10}{100} = 0.1$

$q = 1 - 0.1 = 0.9$

pmf of Bionomial distribution
$P(X=r) = {}^nC_r\, p^r q^{n-r}$ $\qquad r = 0,1,2 \ldots$

(i) $P(X=2) = {}^{12}C_2\, (0.1)^2\, q^{12-2}$

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$$= \frac{12 \times 11}{2 \times 1} \times 0.1 \times 0.1 \times (0.9)^{10}$$

$$= 66 \times (0.01) \times (0.9)^{10} = 0.2301$$

(ii) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - (P(X=0) + P(X=1))$

$= 1 - [{}^{12}C_0 (0.1)^0 (0.9)^{12} + {}^{12}C_1 (0.1)^1 (0.9)^{11}]$

$= 1 - (0.9)^{11} [1 + 12 \times 0.1 \times 0.9]$

$= 1 - (0.9)^{11} (1 + 12 \times 0.09)$

$= 1 - (0.9)^{11} (2.08) = \cancel{0.6527} = 0.3409$

$= 0.341$

iii) $P(X=0) = {}^{12}C_0 (0.1)^0 (0.9)^{12} = (0.9)^{12}$

$= 0.2824$

=

Q. Probability that a man aged 60 would be alive till the 70 years of age is 0.65 find the probability that at least 7 out of 10 such ~~such~~ men would be alive till 70 year of age.

A - X = The man

$n = 10$, $p = 0.65$ $q = 1 - 0.65 = 0.35$

pmf of Bionomial distribution

$$P(X=r) = {}^nC_r \, p^r q^{n-r}$$

i) $P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_7 (0.65)^7 (0.35)^3 + {}^{10}C_8 (0.65)^8 (0.35)^2$

$+ {}^{10}C_9 (0.65)^9 (0.35) + {}^{10}C_{10} (0.65)^{10}$

$$P(X \geq 7) = \frac{\cancel{10}\times \cancel{9}^{3}\times \cancel{8}^{4}}{\cancel{3}\times 2\times 1}(0.65)^7(0.35)^3 + \frac{\cancel{10}^{5}\times 9}{\cancel{2}\times 1}(0.65)^8(0.35)^2$$

$$+ 10\times(0.65)^9(0.35) + 1\times(0.65)^{10}$$

$$= \cancel{120\times 0.04902\times}$$

$$= 0.2522 + 45(0.65)^8(0.1225) + 10\times 0.35(0.65)^9$$
$$+ (0.65)^{10}$$

$$= 0.2522 + 0.1756 + 0.0724 + 0.013$$
$$= 0.5138$$

Q. Fit a binomial distribution to the following data:

x	0	1	2	3	4	5
f	2	14	20	34	22	8

$n=5$

Sol^n

$$np = \frac{\sum_i x_i f_i}{\sum_i f_i}$$

x	f	xf	$P(x) = {}^5C_x p^x q^{5-x}$	$f(x) = N \times P(x)$
0	2	0	${}^5C_0 p^0 (0.432)^5 = 0.01505$	$f(x) = 1.505$
1	14	14	${}^5C_1 (0.568)(0.432)^4 = 0.0989$	$f(x) = 9.89$
2	20	40	${}^5C_2 (0.568)^2 (0.432)^3 = 0.2601$	$f(x) = 26.01$
3	34	102	${}^5C_3 (0.568)^3 (0.432)^2 = 0.3419$	$f(x) = 34.19$
4	22	88	${}^5C_4 (0.568)^4 (0.432)^1 = 0.2248$	$f(x) = 22.48$
5	8	40	${}^5C_5 (0.568)^5 = 0.0591$	$f(x) = 5.91$
	$\sum f = 100$	$\sum fx = 284$		

$$5p = \frac{284}{100} = 2.84$$

$$p = \frac{2.84}{5} = 0.568$$

$q = 1 - 0.568 = 0.432$

frequency distribution as obtained by fitting of Binomial distribution $\Rightarrow$

x	0	1	2	3	4	5
$f(x)$	1.505	9.89	26.01	34.19	22.48	5.91

Poisson Distribution:-

A discrete R.V. X which can take random values 0, 1, 2 --- is said to be follow Poisson distribution if its pmf is given by

$$\boxed{P(X=r) = \frac{e^{-m} m^r}{\lfloor r}} \quad , r = 0, 1, 2 --- \quad m > 0$$

$m \rightarrow$ Parameter of the distribution

$X \sim P(m) \rightarrow$ Poisson variate

* Poisson distribution can be derived as a limiting case of Binomial distribution under the conditions $n \to \infty$, $p \to 0$ and $\boxed{np = m}$

Mean & Varience of Poisson Distribution:-

$$\text{Mean} (\mu) = E(X) = \sum_{r=0}^{\infty} r \cdot P(X=r)$$

$$= \sum_{r=0}^{\infty} r \cdot \frac{e^{-m} m^r}{\lfloor r} = \sum_{r=1}^{\infty} \frac{\cancel{r}\, e^{-m} m^r}{\cancel{r} \lfloor r-1}$$

$$= e^{-m}\left[m + \frac{m^2}{\lfloor 1} + \frac{m^3}{\lfloor 2} + ---\right]$$

$$= e^{-m} \cdot m\left[1 + m + \frac{m^2}{\lfloor 2} + \frac{m^3}{\lfloor 3} + ---\right]$$

$$= e^{-m} \cdot m \cdot e^{m} = m \qquad \boxed{\text{Mean} (\mu) = m}$$

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum_{r=0}^{\infty} r^2 p(r)$$

$$E(x^2) = \sum_{r=0}^{\infty} r^2 \frac{e^{-m} m^r}{\lfloor r}$$

$$= \sum_{r=0}^{\infty} (r(r-1)+r) \frac{e^{-m} m^r}{\lfloor r}$$

$$= \sum_{r=2}^{\infty} \frac{\cancel{r(r-1)}}{\cancel{r(r-1)}} \frac{e^{-m} m^r}{\lfloor r-2} + \sum_{r=1}^{\infty} \frac{\cancel{r}}{\cancel{r}} \frac{e^{-m} m^r}{\lfloor r-1}$$

$$= e^{-m} \sum_{r=2}^{\infty} \frac{m^r}{\lfloor r-2} + e^{-m} \sum_{r=1}^{\infty} \frac{m^r}{\lfloor r-1}$$

$$= e^{-m}\left[m^2 + \frac{m^3}{\lfloor 1} + \frac{m^4}{\lfloor 2} \ldots\right] + e^{-m}\left[m + \frac{m^2}{\lfloor 1} + \frac{m^3}{\lfloor 2}\right]$$

$$= e^{-m}.m^2\left[1 + m + \frac{m^2}{\lfloor 2} \ldots\right] + e^{-m}.m\left[1 + m + \frac{m^2}{\lfloor 2} \ldots\right]$$

$$= e^{-m}.m^2.e^m + e^{-m}.m.e^m$$

$$= m^2 + m$$

$$E(x^2) = m^2 + m$$

Varience $= E(x^2) - (E(x))^2$

$$= \cancel{m^2} + m - \cancel{m^2} = m$$

$$\boxed{\sigma^2 = m}$$

Standard derivate $\boxed{\sigma = \sqrt{m}}$

Fitting of Poisson Distribution $\Rightarrow$

Let $X \sim P(m)$

Recurrence Relⁿ per probabilities of Poisson distribution

$$\boxed{P(r+1) = \frac{m}{r+1}\,P(r)} \quad r = 0,1,2 \cdots$$

Q. Find the probability that atmost 5 defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective.

Soln:- X = no. of defective fuses $= 0,1,2,3,4,5$

$n = 200 \qquad P = 0.02 \qquad , \quad m = 200 \times 0.02 = 4$

$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \frac{e^{-4}(4)^0}{\lfloor 0} + \frac{e^{-4}(4)^1}{\lfloor 1} + \frac{e^{-4}(4)^2}{\lfloor 2} + \frac{e^{-4}(4)^3}{\lfloor 3}$$

$$+ \frac{e^{-4}(4)^4}{\lfloor 4} + \frac{e^{-4}(4)^5}{\lfloor 5}$$

$$= \frac{1}{e^4} + \frac{1}{e^4}\times 4 + \frac{1}{e^4}\times 8 + \frac{1}{e^4}\times\frac{\cancel{64}\ 32}{\cancel{2}\times3\times\cancel{2}\times1}$$

$$+ \frac{1}{e^4}\times\frac{\cancel{4}\times\cancel{64}\ 32}{\cancel{4}\times3\times\cancel{2}\times1} + \frac{1}{e^4}\left(\frac{4\times\cancel{4}\times\cancel{64}\ 32}{5\times\cancel{4}\times3\times\cancel{2}\times1}\right)$$

$$= \frac{1}{e^4}\left(1 + 4 + 8 + \frac{32}{3} + \frac{32}{3} + \frac{128}{15}\right)$$

$$= \frac{1}{e^4}\left(\frac{39+64}{3} + \frac{128}{15}\right) = \frac{1}{e^4}\left(\frac{103}{3} + \frac{128}{15}\right)$$

$$= \frac{1}{e^4}\left(34.34 + 8.53\right) = \frac{42.87}{e^4}$$

$P(X \le 5) = 0.7851$

Q. In a certain factory turning out razer blades there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packet of 10 using poisson distribution find the no. of packets containing

(i) No defective

(ii) one defective

(iii) 2 defective blades respectively in a consignment of 10,000 packet.

Ans

X = no. of defective blades

$n = 10$, $N = 10000$

$p = 0.002$

$m = 0.02$

$$P(X=0) = \frac{e^{-m} m^0}{\lfloor 0} = \frac{e^{-0.02}(1)}{1} = 0.980198$$

no. of packets containing

$10,000 \times 0.9801986$

$= \cancel{9801.986}$ 9801.9 packets

$P(X=1) = e^{-0.02} \times 0.02 = 0.980198 \times 0.02$

$= 0.019603973$

$N \times 0.019603973 = 10000 = 0.019603973$

$= \cancel{1960.39}$ 196.0 packets

$$P(X=2) = \frac{e^{-0.02} \times (0.02)^2}{\lfloor 2} = \frac{e^{-0.02} \times (0.02) \times 0.02}{2}$$

$= 0.000196039$

no. of packets containing 2 blades

$= 10000 = 0.0001960$

$= 1.96$ packets

——x——

Q. 4) Letters are received in an office on each one of 100 days. Assuming the following data to form a random sample from a poisson distribution. Find the expected frequencies correct to nearest unit. (Given $e^{-4} = 0.0183$)

No. of letters (x)	0	1	2	3	4	5	6	7	8	9	10
frequency (f)	1	4	15	22	21	20	8	6	2	0	1

Ans –

x	f	fx	$P(x) = \frac{e^{-m} m^x}{\lfloor x}$	$f(x) = 100 \times P(x)$
0	1	0	$P_0 = e^{-4}(4)^0/\lfloor 0 = e^{-4} = 0.01832$	$1.8 \sim 2$
1	4	4	$P_1 = e^{-4}\, 4/1 = 4/e^4 = 0.7326$	$7.32 \sim 7$
2	15	30	$P_2 = 8/e^4 = 0.14652$	$14.65 \sim 15$
3	22	66	$P_3 = 16 \times 4 \times e^{-4}/3 \times 2 \times 1 = \frac{32}{3} e^{-4} = 0.1953$	$19.5 \sim 20$
4	21	84	$P_4 = 32/3e^4 = 0.1953$	$19.5 \sim 20$
5	20	100	$P_5 = 128/15e^4 = 0.1566$	$15.6 \sim 16$
6	8	48	$P_6 = e^{-4} \times 64 \times 64/6 \times 5 \times 4 \times 3 \times 2 = 256/45e^4 = 0.104$	$10.4 \sim 10$
7	6	42	$P_7 = e^{-4} \times 64 \times 64 \times 4/7 \times 6 \times 5 \times 4 \times 3 \times 2 = 1024/315e^4 = 0.0595$	6
8	2	16	$P_8 = e^{-4} \times 64 \times 64 \times 16/8 \times 6 \times 5 \times 4 \times 3 \times 2 = 512/45e^4 = 0.0297$	3
9	0	0	$P_9 = e^{-4} \times 64 \times 64 \times 64/9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 4 \times 2 \times 1 = 2048/567e^4 = 0.0132$	
10	1	10	$P_{10} = e^{-4} \times 64 \times 64 \times 64 \times 4/10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 0.08529$	1

frequency distribution as obtained by fitting of Poisson distribution:—

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	2	7	15	20	20	16	10	6	3	1	1

NORMAL distribution :—

A continuous R.V. X is said to be follow normal distribution with parameters μ (mean) & σ (S.D) if its pdf is given by

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \quad , \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

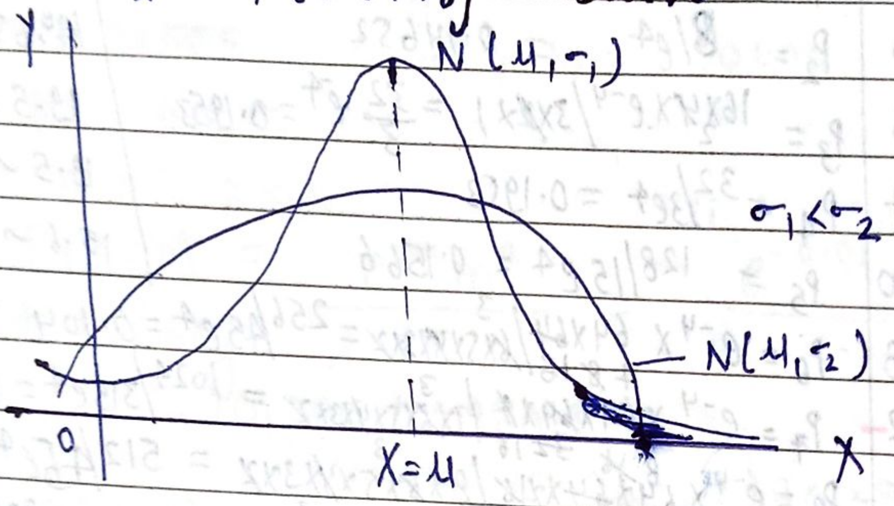
$X \sim N(\mu, \sigma)$ $\longrightarrow$ Normal ~~distribution~~ variate

As $f(x)$ is pdf

i) $f(x) \geq 0$, $\forall -\infty < x < \infty$, $\sigma > 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Normal Probability curve:—



$y = f(x)$ is called Normal Probability curve
(Bell shaped curve)

Properties :—

(i) Bell shaped curve is symmetrical about the mean $x = \mu$

(ii) Mean, median & mode of distribution coincide.

(iii) X - axis is an ~~a~~ ~~cent~~ asymptote.

(iv) Point of inflextions are $\mu \pm \sigma$
if σ is relatively large the curve tends to be flat.

~~A(v)~~ $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow$ The area line under the normal curve is unity

Standard form of the normal distribution:—

The normal curve ~~for~~ $y = f x$ depends on μ & σ. We standardise the normal variate X by using transformation $\boxed{Z = \frac{X-\mu}{\sigma}}$

For $\mu = 0$ & $\sigma = 1$

standard normal variate $Z \sim N(0,1)$

As $f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

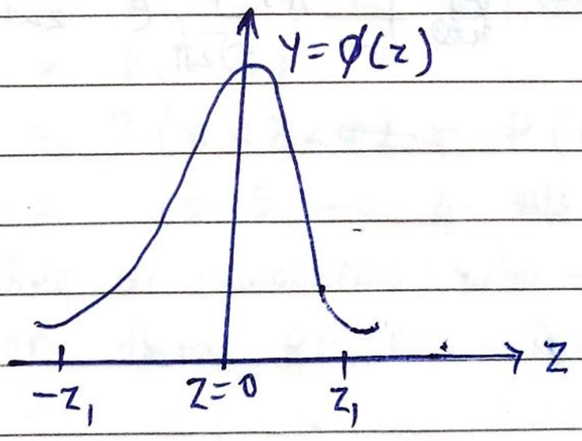
$$Z = \frac{x-\mu}{\sigma}$$

$$\phi(z)\,dz = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

pdf for z

$$\boxed{\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}} \quad , -\infty < z \leq \infty$$

Also bell shaped curve is symmetrical about $Z = 0$



for any $z_1 > 0$

$P(-z_1 \leq Z \leq 0) = P(0 \leq Z \leq z_1)$ (due to symmetry)

To find $p(x_1 < X < x_2)$

obtain $z_1 = \frac{x_1-\mu}{\sigma}$ and $z_2 = \frac{x_2-\mu}{\sigma}$

$$P(x_1 < X < x_2) = P(z_1 < Z < z_2)$$

* Try to convert $P(z_1 < Z < z_2)$ in the form of $P(0 < Z < z_1)$ or $P(0 < Z < z_2)$
* Use normal table to find $P(0 < Z < z)$

Fitting of NORMAL Distribution :-

For a given frequency distribution.

x_i / f_i , $i = 1, 2 --- n$

find $\mu = \dfrac{\sum f_i x_i}{\sum f_i}$

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)^2}$$

$\therefore$ Normal curve fitted to the given data is :-

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}} ; -\infty < x < \infty$$

Q. The distribution of weekly wages for 500 workers in a factories is approximately normal with mean & standard deviation of R.S. 75 & 15 respectively find the no. of workers who recieves weekly wages

(i) More than Rs. 90

(ii) Less than Rs. 45

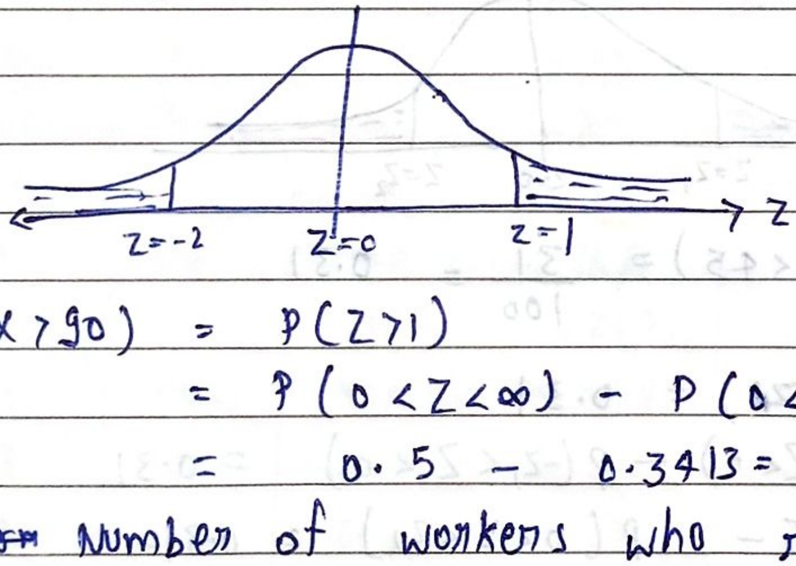
Ans - $X \sim N(\mu, -)$

Let $X \rightarrow$ weekly wages of workers

$\mu = 75, \quad \sigma = 15$

$$Z = \frac{x - \mu}{\sigma}$$

$$Z_1 = \frac{90 - 75}{15} = 1, \quad Z_2 = \frac{45 - 75}{15} = -2$$



(i) $P(X > 90) = P(Z > 1)$

$= P(0 < Z < \infty) - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

$\therefore$ ~~Num~~ Number of workers who receive weekly wages more than from Rs. 90 $= 500 \times P(X > 90)$

$= 500 \times 0.1587$

$= 79.35 \approx 79$ workers

(ii) $P(X < 45) = P(Z < -2)$

$= P(-\infty < Z < 0) - P(-2 < Z < 0)$

$= 0.5 - P(0 < Z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$

∴ Number of workers who receive weekly wages less than 45

$= 500 \times 0.0228 = 11.4 \Rightarrow 11$ workers

Q. In a normal distribution 31% of items are under 45 & 8% are over 64 find the parameters of the distribution.

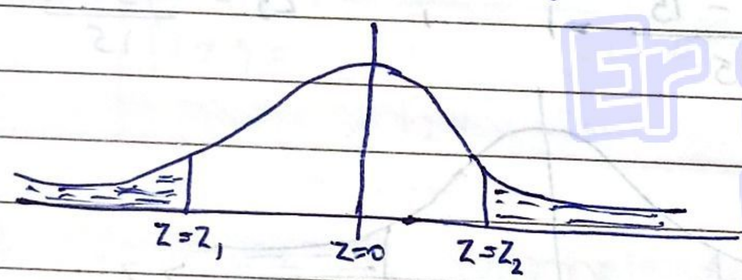
Sol^n:—

$X \sim N(\mu, \sigma)$

$\mu \rightarrow$ mean & $\sigma =$ S.D.

for $x_1 = 45$, $Z_1 = \dfrac{45-\mu}{\sigma}$ ——— (1)

for $x_2 = 64$, $Z_2 = \dfrac{64-\mu}{\sigma}$ ——— (2)



$P(X<45) = \dfrac{31}{100} = 0.31$

$P(Z<-Z_1) = 0.31$

$P(-\infty<Z<0) - P(-Z_1<Z<0) = 0.31$

$0.5 - P(0<Z<Z_1) = 0.31$

$P(0<Z<Z_1) = 0.19$

$\boxed{Z_1 = 0.5}$ (from the table)

$P(X>64) = \dfrac{8}{100} = 0.08$

$P(Z>Z_2) = 0.08$

$P(0<Z<\infty) - P(0<Z<Z_2) = 0.08$

$P(0<Z<Z_2) = 0.42$

$\boxed{Z_2 = 1.41}$

By eqⁿ – (1) & (2)

$$0.5 = \frac{45 - \mu}{\sigma} \quad \text{—— (3)}$$

$$1.41 = \frac{64 - \mu}{\sigma} \quad \text{—— (4)}$$

By eqⁿ – (3) & (4)

$$\mu = 49.9 \sim 50$$

$$\sigma = 9.9 \sim 10$$

$$\boxed{X \sim N(50, 10)}$$

Q. Fit a normal curve to the following frequency distribution.

x :	4	6	8	10	12	14	16	18	20	22	24
y :	1	7	15	22	35	43	38	20	13	5	1

Solⁿ:—

x	f	x^2	fx	fx^2	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$
4	1	16	4	16	
6	7	36	42	252	
8	15	64	120	960	
10	22	100	220	2200	
12	35	144	420	5040	
14	43	196	602	8428	
16	38	256	608	9728	
18	20	364	360	6280	
20	13	400	260	5200	
22	5	484	110	2420	
24	1	576	24	576	
	$\Sigma f = 200$		$\Sigma fx = 2770$	Σfx^2 42,300	

$$\mu = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{2770}{200} = 13.85$$

$$\sigma = \sqrt{\frac{\Sigma f_i x_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i x_i}{\Sigma f_i}\right)^2}$$

$$\sigma = \sqrt{\frac{41300}{200} - (13.85)^2}$$

$$\sigma = \sqrt{206.5 - 191.82} = 3.83$$

$\therefore$ The normal curve to be fitted.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

with $\mu = 13.85$, $\sigma = 3.83$.

Rectangular / Uniform Distribution:-

A continuous R.V. X is said to follow a continuous uniform or rectangular distribution over an interval (a,b) if its probability density function is given by

$$f(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant

$\therefore$ $f(x)$ is a pdf

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$k\int_a^b dx = 1 \Rightarrow k[x]_a^b = 1$$

$$k(b-a) = 1$$

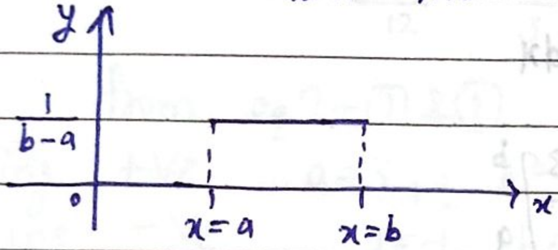
$$\boxed{k = \frac{1}{b-a}}$$

$\therefore$ pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

* Since the curve $y=f(x)$ describes a rectangle over the x-axis and b/w the ordinates $x=a$ & $x=b$

$\therefore$ This distribution is also known as rectangular distribution



Rectangular Distribution Curve

* For a uniform variate X depend in $(-a, a)$ the pdf in

$$f(x) = \begin{cases} \frac{1}{2a}, & -a<x<a \\ 0, & \text{otherwise} \end{cases}$$

Moment, Mean & Variance of Uniform Distribution:-

Moment generating function about the origin

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}.f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}.dx$$

$$M_x(t) = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{b-a}\left(\frac{e^{tb}-e^{ta}}{t}\right)$$

Moment about the origin

$\mu_r' = E[(X-0)^r]$

$$\mu_1' = E(X-0) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x.dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$\frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{1}{\cancel{b-a}}\frac{(b+a)\cancel{(b-a)}}{2}$$

$$\mu_1' = \frac{b+a}{2}$$

$$\boxed{\text{Mean } \mu_1' = \frac{a+b}{2}}$$

$$\mu_2' = E(x-0)^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$\mu_2' = \frac{1}{b-a}\int_a^b x^2\,dx$$

$$\mu_2' = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$\mu_2' = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{\cancel{b-a}}\left(\frac{\cancel{(b-a)}(b^2+ab+a^2)}{3}\right)$$

$$\mu_2' = \frac{a^2+ab+b^2}{3}$$

Variance $\sigma^2 = \mu_2' - (\mu_1')^2$

$$\sigma^2 = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$\sigma^2 = \frac{a^2+ab+b^2}{3} - \frac{(a^2+2ab+b^2)}{4}$$

$$\sigma^2 = \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$\sigma^2 = \frac{a^2-2ab+b^2}{12}$$

Variance $\boxed{\sigma^2 = \frac{(a-b)^2}{12}}$

Standard deriviate $= \boxed{\sigma = \frac{a-b}{2\sqrt{3}}}$

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Q. If X is uniformly distributed with mean 1 & variance $4/3$ find probability $P(X<0)$

Ans - $X \sim U(a,b)$

Mean $=1$

variance $= 4/3$

$$\text{mean} = \frac{a+b}{2} = 1 \Rightarrow a+b=2 \quad \text{—(1)}$$

$$\text{Variance} = \frac{(a-b)^2}{12} = \frac{4}{3} \Rightarrow (a-b)^2 = 16$$

$$a-b = \pm 4 \quad \text{—(2)}$$

from eq^n - (1) & (2)

Taking +ve $a=3$, $b=-1$

Taking -ve $a=-1$, $b=3$

from $a<x<b$

$a=-1$, $b=3$

pdf of X is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} 1/4, & -1<x<3 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X<0) = \int_{-1}^{0} f(x)\,dx = \frac{1}{4}\int_{-1}^{0} dx$$

$$= \frac{1}{4}$$

Q. Find the r^{th} moment of the rectangular distribution about the point $x=a$ whose pdf is $f(x) = \frac{1}{b-a}$ for $a \le x \le b$.

- ~~Moment generating function about the point a~~

~~$M_x(t) = E(e^{t(x-a)}) =$~~

Moment about the point $x = a$

$$\mu_r'' = E[(x-a)^r] = \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b (x-a)^r\,dx$$

$$= \frac{1}{b-a}\left[\frac{(x-a)^{r+1}}{r+1}\right]_a^b = \frac{1}{b-a}\left[\frac{(b-a)^{r+1}}{r+1}\right]$$

$$\boxed{\mu_r'' = \frac{(b-a)^r}{r+1}}$$

Exponential Distribution $\Rightarrow$

A continuous R.V. X is said to follow an exponential distribution. Also called negative exponential distribution with parameter $\lambda > 0$ if its pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

To prove of pdf :-

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} e^{-\lambda x}\,dx$$

$$= \frac{-\cancel{\lambda}}{\cancel{\lambda}}\left[e^{-x}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right] = 1$$

Moments, moment generating function, mean & Variance for exponential distribution:—

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}.f(x)\,dx$$

$$= \lambda\int_0^{\infty} e^{tx}.e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} e^{(t-\lambda)x}\,dx = \lambda\int_0^{\infty} e^{-(\lambda-t)x}\,dx$$

$$M_x(t) = \frac{-\lambda}{\lambda-t}\left[e^{-\infty} - e^{-0}\right]$$

$$M_x(t) = \frac{\lambda}{\lambda-t}(+1) = \frac{\lambda}{\lambda-t}$$

Moment about the origin:—

$$\mu_r' = E(X-0)^r$$

$$\mu_1' = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^{\infty} x\cdot\lambda e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} x\cdot e^{-\lambda x}\,dx$$

$$= \lambda\left[x\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-\lambda x}}{-\lambda}\,dx\right]$$

$$= \lambda\left[\frac{0x}{-\lambda}\left[e^{-\infty} - e^{0}\right] + \frac{1}{\lambda}\left[\frac{e^{-\lambda}}{-\lambda}\right]_0^{\infty}\right]$$

$$= -0(-1) + \left(\frac{-\lambda}{\lambda^2}\right)\left[e^{-\infty} - e^{0}\right]$$

$$\mu_1' = 0 + \frac{1}{\lambda} \Rightarrow \boxed{\text{Mean } \mu_1' = \frac{1}{\lambda}}$$

$$\mu_2' = E(x^2) = \int_0^{\infty} x^2\lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} x^2 e^{-\lambda x}\,dx$$

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$$\mu_2' = \lambda\left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2\int_0^\infty x \frac{e^{-\lambda x}}{-\lambda}\, dx\right]$$

$$\mu_2' = \left[-x^2 e^{-\lambda x}\right]_0^\infty + \frac{2}{\lambda^2}(1)$$

$$\mu_2' = \frac{2}{\lambda^2}$$

Variance $= \sigma^2 = \mu_2' - (\mu_1)^2$

$$\sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

Variance $\sigma^2 = \frac{2\cancel{\lambda}-1}{\cancel{\lambda^2}} \quad \frac{2-1}{\lambda^2} = \frac{1}{\lambda^2}$

Standard derivate $\cancel{\frac{\sqrt{2\lambda-1}}{\lambda}}$ $\boxed{\sigma = \frac{1}{\lambda}}$

<u>r^{th} order moment about origin:—</u>

$$\mu_r' = E(X)^r = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \lambda\int_0^\infty x^r e^{-\lambda x}\,dx \qquad \left\{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx\right\}$$

$$= \int_0^\infty \frac{y^r}{\lambda^r} e^{-y}\,dy \qquad \left\{\begin{array}{l}\text{Let } \lambda x = y\\ \lambda\,dx = dy\end{array}\right\}$$

$$\boxed{\mu_r' = \frac{\Gamma r+1}{\lambda^r} = \frac{\lfloor r}{\lambda^r}}$$

mean $\boxed{\mu_1' = \frac{1}{\lambda}}$

$\boxed{\mu_2' = \frac{2}{\lambda^2}}$ $\boxed{\sigma^2 = \frac{1}{\lambda^2}}$ $\boxed{\sigma = \frac{1}{\lambda}}$

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Memoryless Property of Exponential Distribution ⇒

If X is exponentialy distributed

$$\boxed{P(X > s+t \,/\, X > s) = P(X > t)}, \text{ for any } s, t > 0$$

$$P(X > k) = \int_k^\infty f(x)\,dx$$

$$= \int_k^\infty \lambda e^{-\lambda x}\,dx = e^{-\lambda k}$$

$$\frac{P(X > s+t)}{P(X > s)} = \frac{P(X > s+t \text{ and } X > s)}{P(X > s)}$$

$$= \frac{P(X > (s+t))}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = P(X > t)$$

The converse of above result is also true

∴ If $P(X > s+t \,/\, X > s) = P(X > t)$

then X follows exponential distribution.

Q. The time (in hour) require to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

(i) What is the probability that repaired time (से अधिक) exceeds 2 hours

(ii) What is the conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours.

Ans - given :- $\lambda = 1/2$

~~(i) $P(X > 2) = \int_2^\infty \lambda e^{-\lambda x}\,dx$~~

Let R.V. $X \rightarrow$ time req^n to repair a machine

then pdf of exponential distribution

$$f(x) = \begin{cases} \frac{e^{-1/2 x}}{2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(X>2) = \int_2^\infty f(x)\,dx$

$= \int_2^\infty \frac{e^{-1/2x}}{2}\,dx$

$= \frac{-2}{2}\left[e^{-1/2x}\right]_2^\infty$

$= -1\left[e^{-\infty} - e^{-1}\right] = \frac{1}{e} = 0.3679$

(ii) $P(X>10 / X>9) = P(X>1)$ By memoryless property

$= \int_1^\infty \frac{e^{-1/2x}}{2}\,dx$

$= -\frac{2}{2}\left[e^{-1/2x}\right]_1^\infty$

$= -\left[e^{-\infty} - e^{-1/2}\right]$

$= \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}} = 0.6065$

Q. If x is exponential distributed with parameter λ. Prove that the probability the x exceed its expected value is less than 0.5.

Ans - pdf of E.D is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x>0 \\ 0, & \text{otherwise} \end{cases}$$

$X \sim E(\lambda)$

Mean $= E(X) = \frac{1}{\lambda}$

$P\left(X > \frac{1}{\lambda}\right) < 0.5$

Date : / /

$$P(X > 1/\lambda) \int_{1/\lambda}^{\infty} f(x)\,dx < 0.5$$

$$\int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\,dx < 0.5$$

$$\frac{\lambda}{-\lambda}\left[e^{-\lambda x}\right]_{1/\lambda}^{\infty} < 0.5$$

$$-\left[e^{-\infty} - e^{-\lambda \times \frac{1}{\lambda}}\right] < 0.5$$

$$e^{-1} < 0.5$$

$$\frac{1}{e} < 0.5$$

$$0.3679 < 0.5 \qquad \text{(H.P.)}$$

Q. ~~If x is exp~~ Find the n^{th} central moment of the rectangular distribution

Ans- moment about the central/mean →

$$\mu_n = E((X-\bar{x})^n)$$

$$\mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b (x-\bar{x})^n\,dx = \cancel{\frac{1}{b-a}\left[\left[\frac{x^2}{2}\right]_a^b - \left[\frac{\bar{x}^2}{2}\right]_a^b\right]}$$

$$\mu_n = \frac{1}{b-a}\int_a^b (x-\bar{x})^n\,dx$$

$$\mu_n = \frac{1}{b-a}\left[\frac{(x-\bar{x})^{n+1}}{n+1}\right]_a^b$$

$$\cancel{\mu_n = \frac{1}{b-a}\left[(b-a) - \left(\frac{a+b}{2}\right)\right]^{n+1} \times \frac{1}{n+1}}$$

$$\mu_n = \frac{1}{b-a}\left[\left(b - \frac{a+b}{2}\right)^{n+1} - \left(a - \frac{a+b}{2}\right)^{n+1}\right] \times \frac{1}{n+1}$$

$(-1)^{n+1}\left(\frac{b-a}{2}\right)^{n+1}$

$$\mu_n = \frac{1}{b-a}\left[\left(\frac{b-a}{2}\right)^{n+1} - \left(\frac{a-b}{2}\right)^{n+1}\right] \times \frac{1}{n+1}$$

$$\mu_n = \frac{1}{(b-a)(n+1)2^{n+1}}\left[(b-a)^{n+1}\left(1 - (-1)^{n+1}\right)\right]$$

$$\mu_n = \frac{(b-a)^n}{2^{n+1}(n+1)}\left[1 - (-1)^{n+1}\right]$$

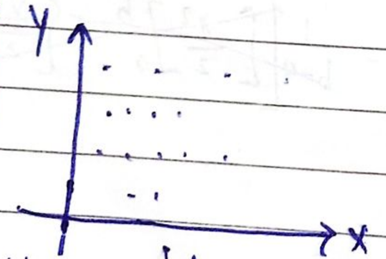
Correlation ⇒ Two variables are said to be correlated if change in one variable gives a specific change in the other variable.

→ If two variables ~~derivate~~ deviate in the same direction then the correlation is said to be direct or positive correlation.

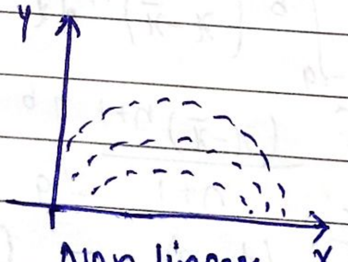
→ If two variables deviate in the opposite direction then the correction is said to be diverse or Negative correlation.

Karl Pearson Coefficient of Correlation :—

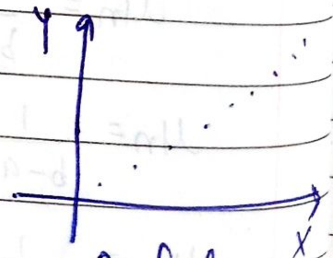
Scatter Diagram



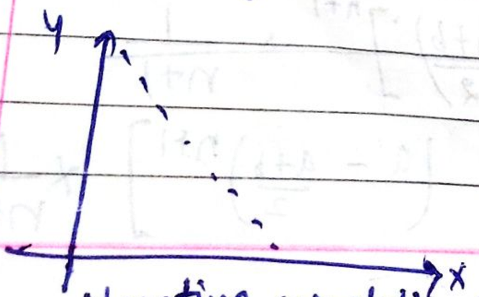
No correlation bet^n x & y



Non linear correlation bet^n x & y



Perfect Positive correlation



Negative correlation bet^n x & y

Karl Pearson gives a formula to measure the intensity & degree of linear relationship b/w two variable, known as correlation coefficient ~~are~~ $r(x,y) = r_{xy} = r$

$$r = r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

* If (x_i, y_i), $i = 1, 2, 3 \ldots n$ is bivariate distribution then

$$\text{cov}(x,y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{n}}, \quad \sigma_y = \sqrt{\frac{\sum_{i=1}^{n}(y-\bar{y})^2}{n}}$$

$$\boxed{r_{xy} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{n}}\sqrt{\frac{\sum_{i=1}^{n}(y-\bar{y})^2}{n}}}}$$

or

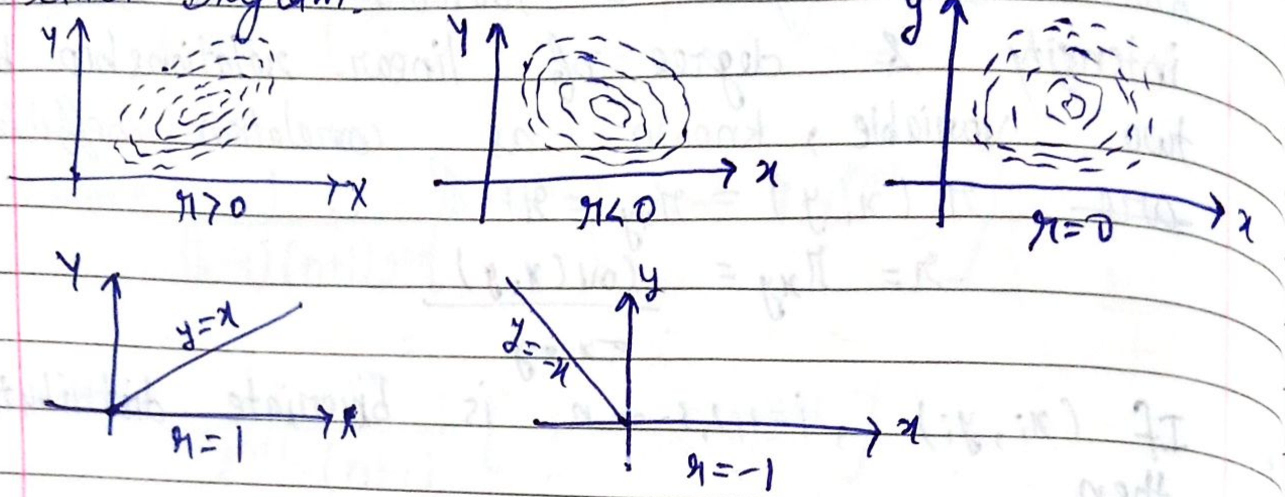
$$\boxed{r_{xy} = \frac{\frac{\sum_{i=1}^{n} x_i y_i}{n} - \bar{x}\bar{y}}{\sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}\sqrt{\frac{\sum y_i^2}{n} - \bar{y}}}}$$

where

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n} x_i y_i}{n} - \bar{x}\bar{y}$$

$$\sigma_x = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}, \quad \sigma_y = \sqrt{\frac{\sum y_i^2}{n} - \bar{y}^2}$$

Scatter Diagram: —



* If $r = \pm 1 \Rightarrow$ A perfect correlation exists bet^n the variable
* If $r = 0 \Rightarrow$ Variables are non correlated
* r_{xy} is a measure of linear relationship bet^n x & y.
* $r = 1 \Rightarrow$ Perfect ~~points~~ positive correlation
* $r = -1 \Rightarrow$ Perfect negative correlation
* $0.75 < |r| < 1 \Rightarrow$ High correlation
* $0.25 < |r| < 0.75 \Rightarrow$ Moderate degree of correlation
* $0 < |r| < 0.25 \Rightarrow$ Low degree of correlation

→ Q. Prove that the coefficient of correlation lie b/w -1 & 1 ie. $-1 \le r_{xy} \le 1$ or $r^2 \le 1$ or

Sol^n:- Let

$$u_i = \frac{x - \bar{x}}{\sigma_x}, \quad v_i = \frac{y - \bar{y}}{\sigma_y}$$

$$\frac{\sum u_i^2}{n} = \frac{\sum_{i=1}^{n} (x - \bar{x})^2}{n\sigma_x^2} = \frac{n\sigma_x^2}{n\sigma_x^2} = 1$$

$$\frac{\sum_{i=1}^{n} v_i^2}{n} = \frac{\sum_{i=1}^{n} (y - \bar{y})^2}{n\sigma_y^2} = \frac{n\sigma_y^2}{n\sigma_y^2} = 1$$

$$\frac{\sum u_i v_i}{n} = \frac{\sum(x-\bar{x})(y-\bar{y})}{n\,\sigma_x\,\sigma_y} = \frac{\mathrm{cov}(x,y)}{\sigma_x\,\sigma_y} = r$$

$\because$

$$\frac{1}{n}\sum(u_i \pm v_i)^2 \geq 0$$

$$\Rightarrow \frac{1}{n}\sum u_i^2 \pm 2u_i v_i + v_i^2 \geq 0$$

$$= \frac{\sum u_i^2}{n} \pm 2\frac{\sum u_i v_i}{n} + \frac{\sum v_i^2}{n} \geq 0$$

$$= 1 \pm 2r + 1 \geq 0$$

$$= 2 \pm 2r \geq 0$$

$$= 1 \pm r \geq 0$$

$$= \pm r \geq 1 \to r \geq 1,\quad r \leq 1$$

$$\Rightarrow \boxed{-1 \leq r \leq 1} \quad (\text{H.P.})$$

Q. Calculate the corelation coefficient for the following height (in inchs) for father(x) and their son(y)

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Ans –

x	y	x^2	y^2	xy
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
$\sum x$ $= 544$	$\sum y =$ 552	$\sum x^2$ $= 37028$	$\sum y^2 =$ 38132	$\sum xy$ $= 37560$

$n = 8$

$\bar{x} = \frac{\Sigma x}{n} = \frac{65+67+66+67+68+69+70+72}{8} = \frac{544}{8} = 68$

$\bar{y} = \frac{\Sigma y}{n} = \frac{67+68+65+68+72+72+69+71}{8}$

$= \frac{552}{8} = 69$

$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} = \sqrt{\frac{37028}{8} - (68)^2} = \sqrt{4.5} = 2.121$

$\sigma_y = \sqrt{\frac{38132}{8} - (69)^2} = \sqrt{5.5} = 2.345$

$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y} = \frac{37560}{8} - 68 \times 69 = 3$

Correlation $= \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{3}{2.121 \times 2.345}$

$r = 0.6032$

there is positive moderalate degree of correlation b/w x & y.

Q. Calculate karl pearson coeff. of correlation of the following data

Sol- $n = 7$

$\bar{x} = \frac{25+27+30+35+33+28+36}{7} = \frac{214}{7} = 30.5$
$\simeq 31$

$\bar{y} = \frac{19+22+27+28+30+23+28}{7} = \frac{177}{7} = 25.2$
$\simeq 25$

180
34
210

$\bar{u} = \frac{\Sigma u}{n} = \frac{-3}{7} = -0.429$

$\bar{v} = \frac{\Sigma v}{n} = \frac{2}{7} = 0.286$

x	y	$u = x - \bar{x}$	$v = y - \bar{y}$	u^2	v^2	uv
25	19	-6	-6	36	36	36
27	22	-4	-3	16	9	12
30	27	-1	2	1	4	-2
35	28	4	3	16	9	12
33	30	2	5	4	25	10
28	23	-3	-2	9	4	6
36	28	5	3	25	9	15
$\Sigma x = 214$	$\Sigma y = 177$	$\Sigma u = -3$	$\Sigma v = 2$	$\Sigma u^2 = 107$	$\Sigma v^2 = 96$	$\Sigma uv = 89$

$$\sigma_u = \sqrt{\frac{\Sigma u^2}{n} - \bar{u}^2} = \sqrt{\frac{107}{7} - (-0.429)^2}$$

$$\sigma_u = \sqrt{15.286 - 0.1840} = \sqrt{15.102} = 3.886$$

$$\sigma_v = \sqrt{\frac{\Sigma v^2}{n} - \bar{v}^2} = \sqrt{\frac{96}{7} - (0.286)^2} = \sqrt{13.71 - 0.081}$$

$$= \sqrt{13.629} = 3.692$$

$$Cov(u, v) = \frac{\Sigma uv}{n} - \bar{u}\bar{v} = \frac{89}{7} + 0.429 \times 0.286$$

$$12.71 + 0.122 = 12.83$$

correlation coeff. $r = \frac{Cov(u,v)}{\sigma_x \sigma_y} = \frac{12.83}{3.886 \times 3.692}$

$$\boxed{r = 0.89}$$

There is positive high correlation betw x & y.

Rank Correlation:— Let a group of n individual grades or ranks (x_i, y_i), $i = 1, 2, \ldots n$ are given with respect to two characteristics A & B resp. then spearman's rank correlation coeff.

~~repeated~~ non repeated rank.

$$\boxed{\rho = 1 - 6\sum d_i^2 / n(n^2-1)}$$

where $d_i = x_i - y_i$

* Rank correlation coeff. for m $(m>1)$ repeated ranks.

If m is the number of times a rank is repeated then $C.F. = \dfrac{m(m^2-1)}{12}$ is to be added to $\sum d_i^2$

$$\boxed{\rho = 1 - \frac{6\left[\sum d_i^2 + C.F.\right]}{n(n^2-1)}}$$

Where C.F. $\rightarrow$ Correlation factor

* $-1 \leq \rho \leq 1$

Q. The ranks of some 10 students in two subjects A & B are given below:—

Ranks in A	5	2	9	8	1	10	3	4	6	7
Ranks in B	10	5	1	3	8	6	2	7	9	4

find rank correlation coeff. for A & B

Sol^n:-

Ranks in A x_i	Ranks in B y_i	$d_i = x_i - y_i$	d_i^2
5	10	-5	25
2	5	-3	9
9	1	8	64
8	3	5	25
1	8	-7	49
10	6	4	16
3	2	1	1
4	7	-3	9
6	9	-3	9
7	4	3	9
			$\Sigma d_i^2 = 216$

$n = 10$

Rank correlation Coeff.

$$\rho = 1 - \frac{6 \Sigma d_i^2}{n(n^2-1)}$$

$$\rho = 1 - \frac{6 \times 216}{10(9)(11)} = 1 - \frac{1296}{990}$$

$\rho = \cancel{1.3091}$

$$\boxed{\rho = -0.3091}$$

Q. Obtain the rank of the correlation coeff. for the following data.

X:	85	74	85	50	65	78	74	60	74	90
Y:	78	91	78	58	60	72	80	55	68	70

dy-

X	Ranks for X x_i	Y	Rank	$d_i = x_i - y_i$	d_i^2
85	2.5	78	3.5	-1.0	1
74	6	91	1	5	25
85	2.5	78	3.5	-1.0	1
50	10	58	9	1	1
65	8	60	8	0	0
78	4	72	5	-1	1
74	6	80	2	4	16
60	9	55	10	-1	1
74	6	68	7	-1	1
90	1	70	6	-5	25
					$\Sigma d_i^2 = 72$

$n = 10$

$$\rho = 1 - \frac{6(\Sigma d_i^2 + C.F.)}{n(n^2-1)}$$

$$\cancel{\rho = 1 - \frac{6 \times 72}{10 \times 9 \times 11} = 1 - \frac{48}{110}}$$

$$\cancel{\rho = 1 - \frac{24}{55}}$$

$$\cancel{C.F. = \frac{m(m^2-1)}{12} = \frac{2.5(6.25-1)}{12}}$$

$$\cancel{C.F. = \frac{2.5 \times 5.25}{12}}$$

$\because$ For $R = 2.5$ repeated twice hence its

$$C.F. = \frac{m(m^2-1)}{12} = \frac{2 \times 3}{12} = \frac{1}{2} = 0.5$$

$\because$ For $R = 6$ ~~repeated~~ twice hence its

$$C.F. = \frac{3 \times 8}{12} = 2$$

$\because$ For $R = 3.5$ repeated twice hence its

$C.F. = 0.5$

total $C.F. = 2+0.5+0.5 = 3$

Rank Correlation Coefficient :—

$$\rho = 1 - \frac{6(\Sigma d_i^2 + C.F.)}{n(n^2-1)}$$

$$\rho = 1 - \frac{6 \times [72+3]}{10(9)(11)} = 1 - \frac{6 \times 75}{10 \times 9 \times 11}$$

$$\rho = 1 - \frac{5}{11} = 1 - 0.454 = 0.546$$

$$\boxed{\rho = 0.546}$$

Curve Fitting ⇒ Curve fitting is the process to find an equation of the curve which fits best to the given observation.

It is useful in infinining the line of regration to estimate the values of one variable which good correspond to the specified value of the other variable.

To determine unique based fits curve is the method of least square.

Principal of least square :— The curve of best fit is that for which errors (residual) or the sum of the squares of the error is minimum.

Fitting of Straight line :—

Let for n observations (x_1, y_1) (x_2, y_2) --- (x_n, y_n) the straight line to be fitted be

$$\boxed{y = a + bx} \quad \text{--- (1)}$$

Error of residual :—

$$e_i = y_i - y_{ci}$$

$$e_i = y_i - (a + bx_i)$$

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (a+bx_i)]^2$$

By the principle of least square, this error must be minimum.

$$\frac{\partial E}{\partial a} = 0 \quad \text{and} \quad \frac{\partial E}{\partial b} = 0$$

$$\frac{\partial E}{\partial a} = -2\sum_{i=1}^{n} (y_i - a - bx_i) = 0$$

$$\boxed{\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i} \quad \text{--- (2)}$$

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0$$

$$\boxed{\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2} \quad \text{--- (3)}$$

Eqn (2) & (3) are known as <u>normal eqn</u> for a & b. By solving the normal eqn, we get a & b which on substitution in eqn (1) obtain the straight line of best fit.

Q. fit a straight line to the following data:

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Ans –

Let straight line to be fitted is

$$y = a + bx \quad \text{—— (1)}$$

By the principle of least squares, normal eq^n are

$$\Sigma y = na + b\Sigma x \quad \text{—— (2)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{—— (3)}$$

$n = 6$

x	y	x^2	xy
1	2.4	1	2.4
2	3	4	6
3	3.6	9	10.8
4	4	16	16
6	5	36	30
8	6	64	48
$\Sigma x = 24$	$\Sigma y = 24$	$\Sigma x^2 = 130$	$\Sigma xy = 113.2$

In eq^n – (2)

$$24 = 6a + 24b$$

100
13.2

$$4 = a + 4b \quad \text{—— (4)}$$

in eq^n – (3)

$$113.2 = a(24) + b(130)$$

$$113.2 = 24a + 130b \quad \text{—— (5)}$$

$$113.2 = 96 - 96b + 130b$$

$$113.2 - 96 = 34b \quad \rightarrow \quad \boxed{b = 0.506}$$

$$\frac{17}{34} = b \quad \Rightarrow \quad \boxed{b = \frac{1}{2}}$$

$\boxed{a = 1.976}$ $\quad$ $\boxed{\cancel{a=2}}$

Then the straight line to be fitted in

$$y = 2 + \frac{1}{2}x$$

$$\boxed{2y = 4 + x}$$

or

$$\boxed{y = 1.976 + 0.506x}$$

Q. fit a straight line to be following data:

x	1	2	3	4	5
y	2	4	6	8	10

x	y	x^2	xy
1	2	1	2
2	4	4	8
3	6	9	18
4	8	16	32
5	10	25	50
$\Sigma x = 15$	$\Sigma y = 30$	$\Sigma x^2 = 55$	$\Sigma xy = 60$

$$\Sigma y = na + b\Sigma x \quad \text{—— (2)}$$
$$30 = 5a + 15b$$
$$6 = a + 3b \quad \text{—— (4)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$
$$60 = 15a + 55b$$
$$12 = 3a + 11b \quad \text{—— (5)}$$

Fitting of Parabola $\Rightarrow$

Let a second degree parabola which is to be fitted to n observations (x_1, y_1), (x_2, y_2) --- (x_n, y_n) be

$$\boxed{y = a + bx + cx^2} \quad \text{--- (1)}$$

then by method of least squares normal eq^n are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad \text{--- (2)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{--- (3)}$$
$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{--- (4)}$$

on solving (2), (3) & (4) we get the values of a, b, c.

Q. Fit a parabola of the second degree to the following data:

x	1	3	5	7	9
y	2	7	10	11	9

Sol^n:- Let eq^n of parabola to be fitted in

$$y = a + bx + cx^2 \quad \text{--- (1)}$$

by the method of least squares normal eq^n:-

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad \text{--- (2)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{--- (3)}$$
$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{--- (4)}$$

$n = 5$

x	y	x^2	xy	x^2y	x^3	x^4
1	2	1	2	2	1	1
3	7	9	21	63	27	81
5	10	25	50	250	125	625
7	11	49	77	539	343	2401
9	9	81	81	729	729	6561
$\Sigma x = 25$	$\Sigma y = 39$	$\Sigma x^2 = 165$	$\Sigma xy = 231$	$\Sigma x^2y = 1583$	$\Sigma x^3 = 1225$	$\Sigma x^4 = 9669$

$$39 = 5a + 25b + 165c \quad \text{--- (2)}$$
$$231 = 25a + 165b + 1225c \quad \text{--- (3)}$$
$$1583 = 165a + 1225b + 9669c \quad \text{--- (4)}$$

$$195 = 25a + 125b + 825c$$
$$\underline{\underset{-}{231} = \underset{-}{25a} + \underset{-}{165b} + \underset{-}{1225c}}$$

$$36 = 40b + 400c$$

$$\begin{array}{r} 231 \\ -195 \\ \hline 36 \end{array}$$

$$9 = 10b + 100c \quad \text{--- (5)}$$
$$1583 = 165a + 1225b + 9669c$$
$$\underline{\underset{-}{1287} = 165a + \underset{-}{825b} + \underset{-}{5445c}}$$

$$296 = 400b + 4224c$$
$$74 = 100b + 1056c \quad \text{--- (6)}$$

Sol^n (5) & (6)

$$90 = 100b + 1000c$$
$$\underline{\underset{-}{74} = \underset{-}{100b} + \underset{-}{1056c}}$$

$$16 = -56c$$
$$c = -\frac{16}{56}$$
$$\boxed{c = -0.286}$$

$$90 = 100b - 286$$
$$376 = 100b$$
$$\boxed{b = 3.757}$$

$$\boxed{a = -1.557}$$

Reg^n eq^n of parabola is

$$\boxed{y = -1.557 + 3.757x - 0.286x^2}$$

Fitting of other Curves ⇒

(i) Fitting of the curve $y = ax^b$ taking log on both side

$\log y = \log a + b \log x$

Let $\log y = Y$, $\log a = A$, $\log x = X$

then $Y = A + bX$

which is eq^n of straight line

(ii) Fitting of curve $y = ab^x$ taking log on both side

$\log y = \log a + x \log b$

Let $\log y = Y$, $\log a = A$, $\log b = B$

then

$Y = A + xB$

which is eq^n of straight line

(iii) $y = ab^{bx}$

$\log y = \log a + bx \log b$

Let $\log y = Y$, $\log a = A$

$Y = A + bx$

Q. Fitt an exponential curve of the form $y = ab^x$ to the following data

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Ans – $y = ab^x$

taking log both side

$$\log y = \log a + x \log b$$

Let $\log y = Y$, $\log a = A$, $\log b = B$

$$n = 8$$

$$Y = A + xB \longrightarrow (1)$$

$$\Sigma Y = nA + B\Sigma x \longrightarrow (2)$$

$$\Sigma xY = A\Sigma x + B\Sigma x^2 \longrightarrow (3)$$

x	y	$Y = \log y$	x^2	$x^2 Y / x \Rightarrow$	xY
1	1	0	1	0	
2	1.2	0.079	4	0.316 / $\cancel{x}$	0.1584
3	1.8	0.255	9	2.295 / $\cancel{x}$	0.7659
4	2.5	0.397	16	6.352 / $\cancel{x}$	1.5916
5	3.6	0.556	25	13.9 / $\cancel{x}$	2.7815
6	4.7	0.672	36	24.192 / $\cancel{x}$	4.0326
7	6.6	0.819	49	40.131 / $\cancel{x}$	5.7365
8	9.1	0.959	64	61.376 / $\cancel{x}$	7.6720
$\Sigma x = 36$	$\Sigma y = 30.5$	$\Sigma Y = 3.7343$	$\Sigma x^2 = 206$	$\Sigma xY \Longrightarrow$ ~~148.562~~	227.385

in eqⁿ (2) $\quad 3.7343 = 8A + B \times 36 \longrightarrow (4)$

$$227.385 = 36A + 206B \longrightarrow (5)$$

Solve these eqⁿ

$$\frac{3.7343 - 36B}{8} = A$$

in eqⁿ (5)

$$227.385 = \overset{9}{\cancel{36}}\left(\frac{3.7343 - 36B}{\cancel{8}_{2}}\right) + 206B$$

$$2 \times 227.385 = 33.6087 - 324B + 412B$$

$454.770 = 33.6087 + 88B$

$\cancel{\frac{421.1613}{88} = B}$

$\cancel{B = 4.7859}$

$B = \frac{11.869}{88}$

$\boxed{B = 0.137}$

$0.467 = A + 4.5B$

22.785

~~$A = -0.148$~~

$\boxed{A = -0.148}$

$\log a = -0.148$, $\boxed{a = 0.711}$

$\log b = 0.137$, $\boxed{b = 1.37}$

Line of Regression ⇒

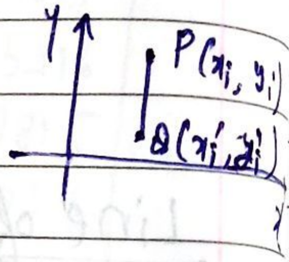
Regression:- A Regression model is a mathematical eqⁿ that describes the relationship b/w two or more variables

Simple Regression:- The impact of a single independent variable on a dependent variable it is called a simple regression and such a model is known as simple regression model.

Multiple Regression:- The impact of two or more independent variable on a dependent variable is known as multiple regression and such a model is known as multiple regression mode.

Line of Regression ⇒ If the curve of Regression is the straight line then it is called a line of regression and the regression is called linear otherwise it is said to be curvy-linear Regression.

(i) Line of Regression y on x :— If the LOR is such that the sum of the squares of the deviation parallel to y axis is minimized



Eqⁿ of LOR

y on x :—

$$y-\bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x-\bar{x})$$

or

$$\boxed{y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})} \qquad \mu_{11} = \sigma_x\sigma_y r$$

where

regression coefficient of y on x

$$\boxed{b_{yx} = r\frac{\sigma_y}{\sigma_x}}$$

(ii) Line of regression x on y :— If the LOR is such that the sum of squares of the deviation parallel to x-axis is minimized.

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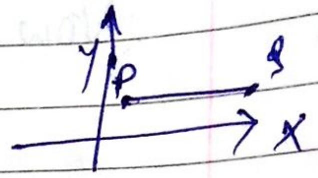
Eqn of LOR :—

x on y :—



$$x-\bar{x} = \frac{\mu_{11}}{\sigma_y^2}(y-\bar{y})$$

or $\qquad \mu_{11} = \sigma_x \sigma_y r$

$$\boxed{x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})}$$

where regression coefficient of x on y

$$\boxed{b_{xy} = r\frac{\sigma_x}{\sigma_y}}$$

→ Correlation coefficients and the two regression coeff. have the same sign.

$$\boxed{b_{xy}\, b_{yx} = r^2}$$

Q. Two R.V. have the least square regression lines with eqn $3x+2y-26=0$ & $6x+y-31=0$ find the mean values & the coeff. of regression betn x & y. and find all correlation coff.

1. Let lines of regression passes through the point $(\bar{x},\bar{y})$

$$3\bar{x}+2\bar{y}-26=0 \quad \text{—— (1)}$$

$$6\bar{x}+\bar{y}-31=0 \quad \text{—— (2)}$$

$$3\bar{x}+2\bar{y}=26$$

$$12\bar{x}+2\bar{y}=62$$

$$-9\bar{x}=-36$$

$$\boxed{\bar{x}=4}$$

$$\bar{y}=\frac{26-12}{2}=\frac{14}{2}$$

$$\boxed{\bar{y}=7}$$

Now by the line of regression:—

$$3x+2y-26=0$$

$$y = \frac{-3}{2}x + 13$$

which is line of regression y on x:—

$$b_{yx} = -\frac{3}{2}$$

Similarly $6x+y-31=0$

$$x = -\frac{y}{6} + \frac{31}{6}$$

which is line of regression x on y:—

$$b_{xy} = \frac{-1}{6}$$

$$\because r^2 = b_{yx} \times b_{xy}$$

$$r^2 = \left(-\frac{3}{2}\right) \times \left(-\frac{1}{6}\right)$$

$$r^2 = \frac{1}{4}$$

$$r = \sqrt{\frac{1}{4}} \Rightarrow \boxed{r = \pm\frac{1}{2}}$$

since b_{xy} & b_{yx} are −ve
that's why $\boxed{r = -\frac{1}{2}}$

Q. Calculate the coeff. of correlation & obtain the line of regression for the following data:

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

Ans –

$n = 9$

x	y	x^2	y^2	xy
1	9	1	81	9
2	8	4	64	16
3	10	9	100	30
4	12	16	144	48
5	11	25	121	55
6	13	36	169	78
7	14	49	196	98
8	16	64	256	128
9	15	81	225	135
$\Sigma x = 45$	$\Sigma y = 108$	$\Sigma x^2 = 285$	$\Sigma y^2 = 1356$	$\Sigma xy = 597$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{45}{9} = 5$$

$$\bar{y} = \frac{108}{9} = 12$$

$$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y} = \frac{597}{9} - 5 \times 12$$

$$Cov(x,y) = 66.33 - 60 = 6.33$$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} = \sqrt{\frac{285}{9} - (5)^2}$$

$$\sigma_x = \sqrt{31.66 - 25} = \sqrt{6.66} = 2.58$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \bar{y}^2} = \sqrt{\frac{1356}{9} - 144}$$

$$\sigma_y = \sqrt{150.6 - 144} = \sqrt{6.6} = 2.58$$

$$r = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{6.33}{2.58 \times 2.58} = 0.95$$

$\boxed{r = 0.95}$

$0.7 \le r \le 1$

there is positive high correlation betw x & y

LOC of regression y on x :-

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$y - 12 = \frac{0.95 \times \cancel{2.88}}{\cancel{2.88}} (x - 5)$$

$$y - 12 = 0.95x - 4.75$$

$$y - 0.95x = \cancel{16.75}\ 7.25$$

$$\boxed{y - 0.95x - \cancel{7}1.25 = 0} \text{ --- (i)}$$

LOC of regression x on y :-

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$x - 5 = \frac{0.95 \times \cancel{2.88}}{\cancel{2.88}} (y - 12)$$

$$x - 5 = 0.95y - 11.4$$

$$\boxed{x - 0.5y + 6.4 = 0} \text{ --- (ii)}$$

Q. Find the angle b/w the two lines of regression & interpret the cases when $r = \pm 1, 0$

Ans - Eq^n of line of regression y on x

$$y - \bar{y} = \frac{r \sigma_y}{\sigma_x} (x - \bar{x})$$

slope $m_1 = r \frac{\sigma_y}{\sigma_x}$ --- (1)

Eqn of line of regression x on y

$$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

$$\text{slope} = \frac{1}{r\frac{\sigma_x}{\sigma_y}} = m_2 \quad \text{--- (2)}$$

angle b/w the two line of regression is

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2}$$

$$\tan\theta = \frac{\frac{1}{r\frac{\sigma_x}{\sigma_y}} - \frac{r\sigma_y}{\sigma_x}}{1+\frac{\sigma_y}{r\sigma_x}\times\frac{r\sigma_y}{\sigma_x}}$$

$$\tan\theta = \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1+\left(\frac{\sigma_y}{\sigma_x}\right)^2}$$

$$\tan\theta = \frac{\frac{\sigma_y - r^2\sigma_y}{r\sigma_x}}{1+\left(\frac{\sigma_y}{\sigma_x}\right)^2} = \frac{\frac{\sigma_y(1-r)(1+r)}{r(\sigma_x)}}{1+\left(\frac{\sigma_y}{\sigma_x}\right)^2}$$

$$\tan\theta = \frac{\sigma_y(1-r^2)}{r\sigma_x} \times \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} = \frac{\sigma_x\sigma_y(1-r^2)}{r(\sigma_x^2+\sigma_y^2)}$$

$$\boxed{\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)} \quad \text{--- (3)}$$

When $r=\pm1$ then $\tan\theta = 0$ $\boxed{\theta=0^\circ}$ ← the line of regression will parallel

When $r=0$ then $\tan\theta=\infty$ $\boxed{\theta=90^\circ}$ ← perpendicular case-I If $r = 0$

they there is no-relationship b/w the two variables and they are indep

on putting $r = 0$ in eq. (3)

$\tan\theta = \infty$

$\theta = \pi/2$

$\therefore$ lines of regression are perpendicular.

case-II If $r = \pm 1$

on putting these value of r in eq. (3)

$\tan\theta = 0$

~~$\theta = 0$~~ $\theta = 0'$

$\because$ lines of regression are conside, the corre is perfect.

Q. Calculate r & obtain line of regression

x	45	55	56	58	60	65	68	70	75	80	8[illegible]
y	51	50	48	60	62	64	65	70	74	82	9[illegible]

Ay - $n = 11$

$\bar{x} = \frac{\Sigma x}{n} = \frac{717}{11} = 65.18$

$\bar{y} = \frac{\Sigma y}{n} = \frac{721}{11} = 65.54$

x	y	x^2	y^2	xy
45	56	2025	3136	2520
55	50	3025	2500	2750
56	48	2688	2304	2688
58	60	3480	3600	3480
60	62	3600	3844	3720
65	64	4225	4096	4160
68	65	4420	4225	4420
70	70	4900	4900	4900
75	74	5625	5476	5550
80	82	6400	6724	6562
85	90	7225	8100	7650
$\Sigma x = 717$	$\Sigma y = 721$	$\Sigma x^2 = 47613$	$\Sigma y^2 = 48905$	$\Sigma xy = 48400$

$$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y} = \frac{48400}{11} - 65.18 \times 65.64$$

$$= 4400 - 4278.41$$

$$= 121.59 = 121.6$$

$$\sigma x = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} = \sqrt{\frac{47613}{11} - (65.18)^2}$$

$$= \sqrt{4328.45 - 4246.47} = \sqrt{81.98} = 9.054$$

$$\sigma y = \sqrt{\frac{\Sigma y^2}{n} - \bar{y}^2} = \sqrt{\frac{48905}{11} - (65.54)^2} = \sqrt{4445.90 - 4295.49}$$

$$= \sqrt{150.4} = 12.26$$

$$r = \frac{121.6}{9.054 \times 12.26} \quad \frac{121.6}{111.0} = 1.0$$

the line of regression

Optimization

Optimization is the process of maximizing or minimizing a desired objective function restricting the solution to the reason where all the given constants are satisfied.

eg - $\text{Max } Z = 2x_1 + 3x_2$

st
$$x_1 + x_2 \leq 2$$
$$2x_1 - x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Q. A Firm manufactures 3 product A, B & C. The profits are Rs. 3, Rs. 2 & Rs. 4 respectively. The firms has 2 machines & ~~below~~ below is the required processing time in minutes for each machines on each product

machine	Product A	B	C
G	4	3	5
H	2	2	4

Machine G & H have 2000 & 2500 machine minutes respectively. The firm must manufacture 100 A's 200 B's & 50 C's but not more than 150 A's setup & L.P. problem to maximize profit.

Formulation:—

Ans — Let x_1, x_2, x_3 be the no. of product A, B & C respectively

$$Z = 3x_1 + 2x_2 + 4x_3$$

$$4x_1 + 3x_2 + 5x_3 \le 2000$$
$$2x_1 + 2x_2 + 4x_3 \le 2500$$

$$100 \le x_1 \le 150$$
$$200 \le x_2 \ge 0$$
$$50 \le x_3 \ge 0$$

Q. A firm can produce 3 type of clothes say A, B & C. 3 kind of wool are required for it say red, green & blue wool. unit of type A ~~clothe~~ needs 2m of red wool & 3m of blue wool. One unit of type B cloth needs 3m of red wool, 2m of green wool & 2m of blue wool. And one unit of type 3 clothes needs 5m of green wool, & 4m of blue wool. The firm has only a stock of 8m of red wool 10m of green wool and 15m of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3, of type B is Rs. 5 or type C cloth is Rs. 4. Formulate the problem for as & to maximize the income from the finished ~~grot~~ cloth.

Ay - Let x_1, x_2 & x_3 is ~~the colour of~~ A, B, C ~~red, green & blue~~ per unit length respectively

$$\cancel{A = 2x_1 + 3x_3}$$
$$\cancel{B = 3x_1 + 2x_2 + 2x_3}$$
$$\cancel{C = 5x_2 + 4x_3}$$

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

	red	blue	green
$A(x_1)$	2	3	0
$B(x_2)$	3	2	2
$C(x_3)$	0	4	5
	8	15	10

$$2x_1 + 3x_2 \le 8$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$2x_2 + 5x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0$$

Q. A farmer has ~~one~~ 100 acres farm self all tom Lettis, ~~radicals~~ radish he can raise. the price he can obtain is Rs. 1/kg of tomato, Rs. 0.75 ~~kg~~ Head of Lettiau and Rs. 2/kg radix. The average yield per acres in 2000 kg of tomatos 3000 head of ~~lettis~~ lettuce and 1000 kg of radicals. Fertilizer is available at Rs. 0.50 /kg and the amount required per acres is 100 kg each tomato and lettuce and 50 kg

(बुआई) (खेती)

for radishes. Labour required ~~soutting~~ sowing, cultivating & harvesting per acres is 5, man days for tomato & radises. And 6 man days for lettuce. A total of 400 man days of labour are available at Rs. 20 per man day. formulate the problem as a L.P. model to maxime the farmer total profit.

Ans - Let x_1, x_2 & x_3 is tomato, Lettuce & radish on acres respectively.

$$\cancel{Z = x_1 + 0.75x_2 + 2x_3}$$

$$Z = 2000x_1 + 2250x_2 + 2000x_3 - (50x_1 + 50x_2 + 25x_3 + 100x_2 + 100x_1 + 120x_3)$$

$$\cancel{5x_1 + 5x_2 + 6x_3 \le 400}$$

$$\cancel{Z = 1850x_1 + 2100x_2 + 2855x_3}$$

or

Let x_1, x_2 & x_3 is ach of tomato, Lettuce & radish respectively.

$$Z = 1\times 2000x_1 + 0.75\times 3000x_2 + 2\times 1000x_3 - [0.50\times 100x_1 + 0.50\times 100x_2 + .50\times 50x_3 + 20(5x_1 + 5x_3 + 6x_2)$$

$$Z = 2000x_1 - 150x_1 + 2250x_2 - 170x_2 + 2000x_3 - 125x_3$$

$$Z = 1850x_1 + 2080x_2 + 1875x_3$$

$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$x_1 + x_2 + x_3 \leq 100$$
$$x_1, x_2, x_3 \geq 0$$

Q. A resourceful decorator manufacture 2 type of lamp say A & B. Both lamp A & B go through two technicians. first the cutter; second finisher. Lamp A requires 2 hour of cutters time and 1 hour of the finisher time. Lamp B requires 1 hour of cutters & 2 hours of finisher time. The cutter has 104 hour and finisher has 76 hours of time each month. Profit of one lamp A is Rs-6 and on lamp B is Rs. 11 assuming that he can sell all that he produced formulate the problem of best return.

Ans— Let x_1 & x_2 are the no. of lamp A & B

$$Z = 6x_1 + 11x_2$$

Cutter $\quad 2x_1 + x_2 \leq 104$

finisher $\quad x_1 + 2x_2 \leq 76$

$$Z = 6(2x_1 + x_1) + 11(x_2 + 2x_2)$$
$$Z = 12x_1 + 6x_1 + 11x_2 + 22x_2$$
$$Z = 18x_1 + 33x_2$$

$$\text{Max} \quad Z = 6x_1 + 11x_2$$
$$x_1, x_2 > 0$$

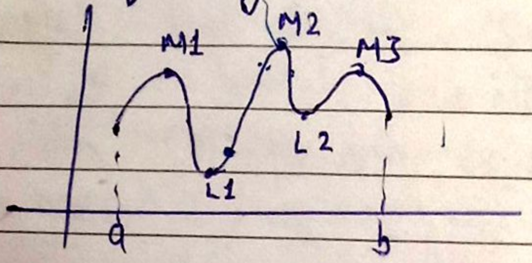
~~Classical optimi~~

For better understanding use NK book

Unit-4

Classical optimization using Differential Calculus

Single variable optimization ⇒ Let a consider a continuous function $f(x)$ of single variable x. In the interval $[a,b]$



M1, M2, M3 → Local maxima or relative maxima
L1, L2 → Local ~~maxim~~ minima
M2 → global maxima
L1 → global minima

→ A function $f(x)$ is said to be <u>Local minima</u> at the point $x = x_0$ if ~~function of~~ $f(x_0) \le f(x_0+h)$ $x_0 \in (a,b)$

→ Local maxima if $f(x_0) \geq f(x_0+h)$
→ global maxima if $f(x_0) \geq f(x) \ \forall \ x[a,b]$
→ global minima if $f(x_0) \le f(x) \ \forall \ x[a,b]$

→ Necessary condition for the single variable function :-

$$f'(x) = 0$$

→ Sufficient condition :- Let $f'(x_0) = f''(x_0) = f'''(x_0)$ —— $f^{n-1}(x_0) = 0$

But $f^{n}(x_0) \neq 0$ then x_0 is

(i) A local minima of $f(x)$, n is even and $f^{n}(x_0) > 0$

(ii) A local maxima of $f(x)$, n is even and $f^{n}(x_0) < 0$

(iii) pointer of inflection, n is odd.

Q. $f(x) = x^2 + 2x + 5$

$f'(x) = 2x + 2 = 0 \Rightarrow x = -1 = x_0$

$f''(x) = 2 > 0$

then $f(x)$ is local minima.

→ if $f''(x) = 0$

then $f'''(x) \neq 0$ then n is odd

of it is pointer of inflection.

Q. Determine the max. or min. value of the function.

$f(x) = 12x^5 - 45x^4 + 40x^3 - 20$

Ans - $f'(x) = 60x^4 - 180x^3 + 120x^2$

Now necessary condition :-

$f'(x) = 0$

$60x^4 - 180x^3 + 120x^2 = 0$

$x^4 - 3x^3 + 2x^2 = 0$

$x^4 - 2x^3 - x^3 + 2x^2 = 0$

$x^3(x-2) - x^2(x-2) = 0$

$(x-2)(x^2)(x-1) = 0$

then

$x - 2 = 0, \quad x^2 = 0, \quad x - 1 = 0$

$x = 0, 0, 2, 1 = x_0$

Now sufficient condition:—

$f''(x) = 240x^3 - 540x^2 + 240x$

if $x=0$, $f''(0) = 0$ it is pt of inflection

if $x=1$, $f''(1) = 480-540$

$f''(1) = -60 < 0$

then $f(x)$ is local max.

if $x=2$, $f''(2) = 240 \times 8 - 540 \times 4 + 240 \times 2$

$f''(2) = 240 > 0$

then on $x=2$ min.

Now $f(x) = 12x^5 - 45x^4 + 40x^3 - 20$

$f(1) = 12 - 45 + 40 - 20$

$f(1) = -13$

So $\boxed{f_{max}(1) = -13}$

Now $f(2) = 12 \times 32 - 45 \times 16 + 40 \times 8 - 20$

$\boxed{f_{min}(2) = -36}$

Q. Determine the min & max. value of function.

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

Ans- $f'(x) = 5x^4 - 20x^3 + 15x^2$

$f'(x) = 0$

$5x^4 - 20x^3 + 15x^2 = 0$

$\cancel{5x^3(x-4) + (15x+1)(15x-1)} = 0$

$5x^4 - 15x^3 - 5x^3 + 15x^2 = 0$

$5x^3(x-3) - 5x^2(x-3) = 0$

$5x^2(x-1)(x-3) = 0$

then

$x = 0, 0, 1, 3 = x_0$

Sufficient condition:—

$f''(x) = 20x^3 - 60x^2 + 30x$

$f''(0) = 0$ it is pt of inflection

if $x = 1$, $f''(1) = 20 - 60 + 30$

$f''(1) = -10 < 0$

the $f(x)$ is local max.

if $x = 3$ $f''(3) = 20 \times 27 - 60 \times 9 + 30 \times 3$

$f''(3) = 540 - 540 + 90$

$f''(3) = 90 > 0$

the $f(x)$ is local min.

Now $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f(1) = 1 - 5 + 5 - 1$

$\boxed{f_{max}(1) = 0}$

Now $f(3) = 243 - 405 + 135 - 1$

$378 - 405$

$f(3) = -27$

$\boxed{f_{min}(3) = -27}$

Multivariable optimization without constraint

$\Rightarrow$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$A_1 = |a_{11}|$, $A_2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(i) Positive definite matrix:— The matrix A will be positive definite if and only if all the value of A_1, A_2 upto A_n are positive.

(ii) Negative definite matrix :- The matrix A will be negative definite if and only if the sign of A_j is $(-1)^j$ For $j = 1, 2, 3 \ldots n$.

i.e. $A_1, A_3, A_5 \ldots$ are negative and $A_2, A_4, A_6 \ldots$ are positive.

(iii) Positive semi definite matrix :- If sum of the A_j are positive & remaining A_j are zero then the matrix is positive semi definite.

(iv) Negative semi definite :- If sum of the A_j are of the sign of $(-1)^j$ are remaining A_j are zero, then the matrix is -ve semi definite.

Indefinite matrix :- Matrix which ~~may~~ doesn't follow the above definiteness rule is called <u>indefinite matrix</u>.

Q. Check the nature of following matrix.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$A_1 = |3| = 3 > 0$

$A_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 > 0$

$$A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{vmatrix} = 3(15-1) - 1(5-1) - 1(-1+3)$$

$$42 - 4 - 2$$

$$= 36 > 0$$

So the nature of A matrix is +ve definite mat

Q. Check the nature of following matrix

(i) $A = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -3 \end{bmatrix}$

(i) $A = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}$

$A_1 = |-12| = -12 < 0$

$A_2 = \begin{vmatrix} -12 & 6 \\ 6 & -3 \end{vmatrix} = 36 - 36 = 0$

this matrix A is -ve semi definite.

(ii) $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

$A_1 = |2| = 2$

$A_2 = [2-9] = -7$ indefinite matrix.

(iii) $A_1 = |-1| = -1$

$A_2 \begin{vmatrix} -1 & -1 \\ -1 & -2 \end{vmatrix} = 2 - 1 = 1$

$A_3 = \begin{vmatrix} -1 & -1 & -1 \\ -1 & -2 & -3 \\ -1 & -2 & -3 \end{vmatrix} = -1(6-4) + 1(3-2) + -1(2-2)$

$-2 + 1 = -1$

So A is -ve definite matrix.

Multivariable optimization:—

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0 \Rightarrow x, y = ?$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}, \quad H_1 = \frac{\partial^2 f}{\partial x^2}$$

$$H_2 = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = 0$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right) \begin{matrix} < 0 & \text{max.} \\ > 0 & \text{min} \end{matrix} \qquad \partial = \frac{\partial^2 f}{\partial x^2}$$

(i)	$H_1 > 0$,	$H_2 > 0$	min
(ii)	$H_1 < 0$	$H_2 > 0$	max.
(iii)	$H_1 \lessgtr 0$	$H_2 < 0$	saddle
(iv)	$H_2 = 0$		need for

Q. $f = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

Obtain the exercise point of $f(x) = ?$

Ans – $\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1$ —— (1)

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 \text{ —— (2)}$$

$$3x_1^2 + 4x_1 = 0 \Rightarrow x_1(3x_1 + 4) = 0$$

$$\Rightarrow x_1 = 0, \quad x_1 = \frac{-4}{3}$$

$$3x_2^2 + 8x_2 = 0 \Rightarrow x_2(3x_2 + 8) = 0$$

$$\Rightarrow x_2 = 0, \quad x_2 = \frac{-8}{3}$$

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$\Rightarrow \quad (0,0) \quad (0, -8/3) \quad (-4/3, 0) \quad (-\frac{4}{3}, -\frac{8}{3})$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\partial x_1 = 3x_1^2 + 4x_1$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0, \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

$$H = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

At $(0,0)$ $\quad H = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$

$H_1 = 4 \qquad H_2 = \begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix} = 32$

$H1 > 0, \quad H2 = 32 > 0$

$(0,0)$ min

At $(0, -8/3)$ $\quad H = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$

$H1 = 4 > 0, \quad H_2 = \begin{vmatrix} 4 & 0 \\ 0 & -8 \end{vmatrix} = -32 < 0$

At $(-4/3, -8/3)$ $\quad H = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$

$H_1 = -4 < 0, \quad H_2 = 32 > 0$

then $\left(\frac{-4}{3}, -\frac{8}{3}\right)$ max.

$f_{min} = 0+0+0+0+6 = 0$

$f_{max} = \left(-\frac{4}{3}\right)^3 + \left(-\frac{8}{3}\right)^3 + 2\left(\frac{-4}{3}\right)^2 + 4\left(\frac{-8}{3}\right)^2 + 6$

At $\left(-\frac{4}{3}, 0\right)$, $H = \begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix}$ $H_1 = -4 < 0$

$H_2 = -32 < 0$ saddle }

Q. $f = 20x_1 - 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$

Ans- $\frac{\partial f}{\partial x_1} = 20 - 4x_2 - 8x_1$ — (1)

$\frac{\partial f}{\partial x_2} = -26 + 4x_1 - 6x_2$ — (2)

$20 = 4x_2 + 8x_1$

$2x_1 + x_2 = 5$ — (3)

$6x_2 - 4x_1 = -26$

$3x_2 - 2x_1 = -13$ — (4)

$-2x_1 + 3x_2 = -13$

$2x_1 + x_2 = 5$

$4x_2 = -8$ $\boxed{x_2 = -2}$

$\boxed{x_1 = \frac{7}{2}}$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$ $H_1 = \frac{\partial^2 f}{\partial x_1^2} = -8 < 0$

$H_2 = \begin{vmatrix} -8 & -4 \\ 4 & -6 \end{vmatrix}$

$H_2 = 42 + 16 = 58 > 0$

Q. Determine the relative max & min of the following function

$$f(x) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$$

Ans - $\frac{\partial f}{\partial x_1} = 1 - 2x_1 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}}$

$\frac{\partial f}{\partial x_2} = x_3 - 2x_2 = 0 \Rightarrow \boxed{2x_2 = x_3}$

$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 = 0 \Rightarrow 2 + x_2 = 2x_3$

$2 + x_2 = 4x_2$

$\boxed{x_3 = \frac{4}{3}}$ $\boxed{\frac{2}{3} = x_2}$

$x_1 = \frac{1}{2}, \quad x_2 = \frac{2}{3}, \quad x_3 = \frac{4}{3}$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$H_1 = -2 < 0$

$H_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$ (max.)

$H_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = (-2)(4-1) = -6 < 0$

So the given point gives maximum value of the function (negative definition condition)

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Solution by the method of constraint variation:—

Necessary condition ⇒

$$J\left(\frac{f, g_1, g_2 \cdots g_m}{x_K, x_1, x_2 \cdots x_m}\right) = \begin{vmatrix} \frac{\partial f}{\partial x_K} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \\ \frac{\partial g_1}{\partial x_K} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots \\ \frac{\partial g_m}{\partial x_K} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix}$$

$K = m+1$

$$J\left(\frac{g_1 \quad g_2 \quad g_3 \cdots g_m}{x_1 \quad x_2 \cdots x_m}\right) \neq 0$$

Q. Minimise $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ condition

$g_1(x) = x_1 - x_2 = 0$ — (1) $\quad g_2(x) = x_1 + x_2 + x_3 - 1 = 0$ — (2)

~~Ag~~ $J\left(\frac{g_1\, g_2 \cdots g_m}{x_1\, x_2 \cdots x_m}\right) \neq 0$

$$J\left(\frac{g_1\, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1-(-1)$$

$$= 2 \neq 0$$

$$J\left(\frac{f\ g_1\ g_2}{x_3\ x_1\ x_2}\right) = \begin{vmatrix} \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_3} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} x_3 & x_1 & x_2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$x_3(1-(-1)) - x_1(0+1) + x_2(0-1)$

$2x_3 - x_1 - x_2 = 0$ — (3)

from eq^n-(1)

$x_1 = x_2$ — (4) $\quad$ ~~$2x_1 - 1 = x_3$~~

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$2(2x_1-1) \cancel{- x_1 - x_1} = 0$

$4x_1 - 2 \cancel{- 2x_1} = 0$

$\cancel{2x_1}$

$x_1 + x_2 \cancel{\;} = x_3 - 1$

$2x_3 + x_3 - 1 = 0$

$3x_3 = 1 \Rightarrow \boxed{x_3 = 1/3}$

$2x_2 = \frac{2}{3} \Rightarrow \boxed{x_2 = \frac{1}{3}}$

$\boxed{x_1 = \frac{1}{3}}$ so $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ Ay

$f_{min} = \frac{1}{2}\left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{1}{6}$ Ay

Q. Min. $f(y) = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2 + y_4^2)$

Such that $g_1(y) = y_1 + 2y_2 + 3y_3 + 5y_4 - 10 = 0$ — ①

$g_2(y) = y_1 + 2y_2 + 5y_3 + 6y_4 - 15 = 0$ — ②

Ay — $J\left(\frac{g_1 \; g_2}{y_1 \; y_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0$

$J\left(\frac{g_1 \; g_2}{y_3 \; y_4}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial y_3} & \frac{\partial g_1}{\partial y_4} \\ \frac{\partial g_2}{\partial y_3} & \frac{\partial g_2}{\partial y_4} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 5 & 6 \end{vmatrix}$

$= 18 - 25 = -7 \neq 0$

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$$J\left(\frac{f \; g_1 \; g_2}{y_2 \; y_3 \; y_4}\right) = \begin{vmatrix} \cancel{\frac{\partial f}{\partial y_1}} & \frac{\partial f}{\partial y_2} & \frac{\partial f}{\partial y_3} & \frac{\partial f}{\partial y_4} \\ \cancel{\frac{\partial g_1}{\partial y_1}} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} & \frac{\partial g_1}{\partial y_4} \\ \cancel{\frac{\partial g_2}{\partial y_1}} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} & \frac{\partial g_2}{\partial y_4} \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{y_1} & y_2 & y_3 & y_4 \\ \cancel{1} & 2 & 3 & 5 \\ \cancel{1} & 2 & 5 & 6 \end{vmatrix}$$

$$= y_2(18-25) - y_3(12-10) + y_4(10-6) = 0$$

$$-7y_2 - 2y_3 + 4y_4 = 0$$

$$4y_4 = 2y_3 + 7y_2 \quad \text{—— (3)}$$

$$J\left(\frac{f \; g_1 \; g_2}{y_1 \; y_3 \; y_4}\right) = \begin{vmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_3} & \frac{\partial f}{\partial y_4} \\ \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_3} & \frac{\partial g_1}{\partial y_4} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_3} & \frac{\partial g_2}{\partial y_4} \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_3 & y_4 \\ 1 & 3 & 5 \\ 1 & 5 & 6 \end{vmatrix} \qquad y_1(18-25) - y_3(6-5) + y_4(5-3)$$

$$-7y_1 - y_3 + 2y_4$$

$$2y_4 = y_3 + 7y_1 \quad \text{—— (4)}$$

from eqn (3) & (4)

$$4y_4 = 2y_3 + 14y_1$$

$$4y_4 = 2y_3 + 7y_2$$

$$- \quad\; - \quad\; -$$

$$14y_1 = 7y_2 \longrightarrow \boxed{2y_1 = y_2}$$

in eq (1) & (2)

$$y_1 + 2y_2 + 3y_3 + 5y_4 = 10$$
$$5y_1 + 3y_3 + 5y_4 = 10$$
$$\underline{\cancel{5y_1} \pm 5y_3 \pm 6y_4 = 15}$$

$2y_1 = y_2$ —— (5)

$$-2y_3 - y_4 = -5$$
$$2y_3 + y_4 = 5 \quad \text{—— (6)}$$
$$2y_4 - y_3 = 7y_1$$
$$-2y_3 + 4y_4 = 14y_1$$
$$\underline{2y_3 + y_4 = 5}$$
$$5y_4 - 14y_1 = 5$$

$$\cancel{4y_4 = 2y_3 + 14y_1}$$
$$\cancel{4y_4 = 2y_3 + 14y_1}$$

In eq^n (4)

$$2y_4 = y_3 + 7\left(\frac{10 - 3y_3 - 5y_4}{5}\right)$$
$$10y_4 = 5y_3 + 70 - 21y_3 - 35y_4$$
$$45y_4 = -16y_3 + 70$$
$$45y_4 + 16y_3 = 70$$
$$\underline{8y_4 \pm 16y_3 = 40}$$
$$37y_4 = 30$$

$$y_3 = \frac{110}{56}$$ $$\boxed{y_4 = \frac{30}{37}}$$

$$2y_3 = 5 - \frac{70}{37} = \frac{285 - 70}{53}$$

$$\boxed{y_4 = \frac{45}{53}}$$

$$2y_3 = \frac{185 - 30}{37} = \frac{155}{37}$$

$$\boxed{y_3 = \frac{155}{74}}$$

5

$$\frac{4 \times 30}{37} = 2 \times \frac{155}{74} + 7 \times y_2$$

$$\frac{120-155}{37 \times 7} \quad \frac{120 - \cancel{310}}{37} = 7 \times y_2$$

$$\frac{-35\cancel{5}}{\cancel{370} \times \cancel{7}} \Rightarrow \boxed{y_2 = \frac{-5}{37}}$$

$$\cancel{y_1 = \frac{-10}{37}}$$

$$\boxed{y_1 = \frac{-5}{74}}$$

(3.) Lagrange method:-

$$f = x_1, x_2 - - - x_n$$

$$g = x_1, x_2 - - - x_m$$

$$\boxed{L = f + \lambda_i g_i}$$

$$\frac{\partial L}{\partial x_n} = 0 \text{ - - - - - },\quad \frac{\partial L}{\partial \lambda_i} = 0 \text{ - - - - } x_1$$

$$\begin{vmatrix}
\frac{\partial^2 L}{\partial x_1^2} - K & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\
\frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - K & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_2} \\
\vdots & & & & \vdots & & & \\
\frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} - K & \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \\
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} & 0 & 0 & \cdots & 0 \\
\vdots & & & & \vdots & & & \\
\frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & & \frac{\partial g_m}{\partial x_n} & 0 & \cdots & \cdots & 0
\end{vmatrix} = 0$$

Q. $\max \ z = 4x_1 - x_1^2 + 8x_2 - x_2^2$

s.t. $x_1 + x_2 = 2 \quad , x_1, x_2 \geq 0$

A -

$$L(x_1, x_2, \lambda) = 4x_1 - x_1^2 + 8x_2 - x_2^2 + \lambda(x_1 + x_2 - 2) \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 + \lambda = 0 \Rightarrow 4 + \lambda = 2x_1 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_2} = 8 - 2x_2 + \lambda = 0 \Rightarrow 8 + \lambda = 2x_2 \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 - 2 = 0 \Rightarrow x_1 + x_2 = 2 \quad \text{--- (4)}$$

$$4 + 8 + 2\lambda = 2(x_1 + x_2)$$

$$12 + 2\lambda = 4$$

$$2\lambda = -8$$

$$\boxed{\lambda = -4}$$

$$4 - 4 = 2x_1 \Rightarrow \boxed{x_1 = 0}$$

$$\boxed{x_2 = 2}$$

Sufficient condition.

$$\begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - K & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial g}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - K & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2-K & 0 & 1 \\ 0 & -2-K & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$-2-k\,(0-1) + 0 + 1(0-(-)(2+k))$

$k+2 + 2+k = 0$

$\boxed{k=-2} < 0$ maximum

so

$Z_{max} = 4(0) - (0)^2 + 8(2) - (2)^2$

$Z_{max} = 16-4$

$\boxed{Z_{max} = 12}$

Q. optain the extreme point of the function
$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$
s.t. $x_1^2 + x_2^2 + x_3^2 = 1$

Ay -

$L(x_1, x_2\, \lambda) = x_1 + x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$ —— (1)

$\frac{\partial L}{\partial x_1} = 1 + 2x_1\lambda = 0$ —— (2) $\Rightarrow x_1 = \frac{-1}{2\lambda}$

$\frac{\partial L}{\partial x_2} = 1 + 2x_2\lambda = 0$ —— (3) $\Rightarrow x_2 = \frac{-1}{2\lambda}$

$\frac{\partial L}{\partial x_3} = 1 + 2x_3\lambda = 0$ —— (4) $\Rightarrow x_3 = \frac{-1}{2\lambda}$

$\frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + x_3^2 - 1 = 0$ —— (5)

$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$

$\frac{3}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{3}{4}$

$\lambda = \pm\frac{\sqrt{3}}{2}$

$x_1 = \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, $x_3 = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

sufficient condition

$$\begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2}-K & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} & \frac{\partial g}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2}-K & \frac{\partial^2 L}{\partial x_2 \partial x_3} & \frac{\partial g}{\partial x_2} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2}-K & \frac{\partial g}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2\lambda-K & 0 & 0 & 2x_1 \\ 0 & 2\lambda-K & 0 & 2x_2 \\ 0 & 0 & 2\lambda-K & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{vmatrix} = 0$$

$$= (2\lambda-K)\begin{vmatrix} \cancel{2\lambda-K} & \cancel{0} & \cancel{0} \\ 0 & 2\lambda-K & 0 \\ 0 & 0 & 2\lambda-K \\ 2x_1 & 2x_2 & 2x_3 \end{vmatrix} -0+0 - 2x_1 \begin{vmatrix} 0 & 0 & 2x_1 \\ 2\lambda-K & 0 & 2x_2 \\ 0 & 2\lambda-K & 2x_3 \\ 2x_2 & 2x_3 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2\lambda-K & 0 & 0 \\ 0 & 2\lambda-K & 0 \\ 0 & 0 & 2\lambda-K \end{vmatrix} = \cancel{(2\lambda-K)(4\lambda^2-4\lambda K+K^2)}$$

$K = 2\lambda$

$$= \cancel{(2\lambda-K)}\left[\cancel{(2\lambda-K)}\right.(2\lambda-K)\begin{vmatrix} 2\lambda-K & 0 & 2x_2 \\ 0 & 2\lambda-K & 2x_3 \\ 2x_2 & 2x_3 & 0 \end{vmatrix} -0+0-2x_1 \begin{vmatrix} 0 & 2\lambda-K & 0 \\ 0 & 0 & 2\lambda-K \\ 2x_1 & 2x_2 & 2x_3 \end{vmatrix}$$

$$(2\lambda-k)\left[(2\lambda-k)(0-4x_3^2)-0+2x_2(0-4x_2\lambda+2kx_2)\right]$$
$$-2x_1\left[-(2\lambda-k)(0-4x_1\lambda+2kx_1)\right]=0$$

$$(2\lambda-k)\left[-8x_3^2\lambda+4kx_3^2-8x_2^2\lambda+4kx_2^2\right]$$
$$+(2\lambda-k)\left[4kx_1^2-8x_1^2\lambda\right]=0$$

$$(2\lambda-k)\left[4kx_1^2-8x_1^2\lambda-8x_3^2\lambda-8x_2^2\lambda+4kx_3^2+4kx_2^2)\right]=0$$

$$(2\lambda-k)\left[4k(x_1^2+x_2^2+x_3^2)-8\lambda(x_1^2+x_2^2+x_3^2)\right]=0$$

$$(4k-8\lambda)(2\lambda-k)\left[x_1^2+x_2^2+x_3^2\right]=0$$

$$\because\ x_1^2+x_2^2+x_3^2=1$$

$$(4k-8\lambda)(2\lambda-k)=0$$
$$k=2\lambda$$
$$k=2x\pm\frac{\sqrt{3}}{2}$$

$$\boxed{k=\pm\sqrt{3}}$$

$k=\sqrt{3}>0$

$\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ are minimum point

$$f_{min}=\frac{-3}{\sqrt{3}}=-\sqrt{3}$$

$$\boxed{f_{min}=-\sqrt{3}}$$

$k=-\sqrt{3}<0$

$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ are maximum point.

$$\boxed{f_{max}=\sqrt{3}}$$

Q. Find the dimension of a cylindrical (with top & bottom) made of sheet metal to maximize its value such that the total surface area is equal to 24π.

A - Max $V = \pi x_1^2 x_2$

$$g = 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi = 0$$

$$L = V + \lambda g$$

$$L = \pi x_1^2 x_2 + \lambda[2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi]$$

$$\frac{\partial L}{\partial x_1} = 2\pi x_1 x_2 + \lambda[4\pi x_1 + 2\pi x_2] = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial x_2} = \pi x_1^2 + \lambda 2\pi x_1 = 0 \quad \text{— (2)}$$

$$\frac{\partial L}{\partial \lambda} = 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi = 0 \quad \text{— (3)}$$

From eqn — (1)

$$2\pi x_1 x_2 = -\lambda 2\pi (2x_1 + x_2)$$

$$\lambda = \frac{-x_1 x_2}{2x_1 + x_2} \quad \text{— (4)}$$

Now from eqn — (2)

$$\cancel{\pi} x_1^2 = -2\lambda \cancel{\pi} x_1$$

$$\lambda = \frac{-x_1}{2} \quad \text{— (5)}$$

From eqn — (4) & (5)

$$\frac{-\cancel{x_1} x_2}{2x_1 + x_2} = \frac{-x_1}{2}$$

$$2x_2 = 2x_1 + x_2$$

$$x_2 = 2x_1 \quad \text{— (6)}$$

Now put these value in eq^n - (3)

$$2\pi x_1^2 + 4\pi x_1^2 - 24\pi = 0$$
$$x_1^2 = 4$$
$$x_1 = \pm 2$$

where $x_1 = 2$ & where $x_1 = -2$

$x_2 = 4$ $\quad$ $x_2 = -4$

& $\lambda = -1$ $\quad$ & $\lambda = 1$

point $(2,4)$ $\quad$ point $(-2,-4)$

Sufficient condition

$$\begin{vmatrix} \frac{\partial^2 L}{\partial x_1^2} - k & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial g}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} - k & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2\pi x_2 + 4\pi\lambda - k & 2\pi x_1 + 2\pi\lambda & 4\pi x_1 + 2\pi x_2 \\ 2\pi x_1 + 2\pi\lambda & -k & 2\pi x_1 \\ 4\pi x_1 + 2\pi x_2 & 2\pi x_1 & 0 \end{vmatrix} = 0$$

For point $(2,4)$ & $\lambda = -1$

$$\begin{vmatrix} 8\pi - 4\pi - k & 4\pi - 2\pi & 8\pi + 8\pi \\ 4\pi - 2\pi & -k & 4\pi \\ 8\pi + 8\pi & 4\pi & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4\pi - k & 2\pi & 16\pi \\ 2\pi & -k & 4\pi \\ 16\pi & 4\pi & 0 \end{vmatrix} = 0$$

$$(4\pi - k)(-16\pi^2) - 2\pi(0 - 64\pi^2) + 16\pi[8\pi^2 + 16\pi k] = 0$$
$$-64\pi^3 + 16k\pi^2 + 128\pi^3 + 128\pi^3 + 256k\pi^2 = 0$$
$$272k\pi^2 + 192\pi^3 = 0$$
$$k = \frac{-12}{17}\pi < 0 \quad \text{maximum}$$

Date :

$$\boxed{f_{max} = 16\pi}$$

→ Multiple variable optimization with inequality constraint (kuhn - Tucker) KT condition ⇒

If the problem is minimize

Min $f(x)$ such that

$g_1(X) \leq 0$, $x_j = 1, 2 --- m$

$X = (x_1 \; x_2 --- x_n)^T$

(I) $$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_J \frac{\partial g_i}{\partial x_j} = 0$$

(II) $\lambda_i g_J = 0$

(III) $g_J \leq 0$

(IV) $\lambda_J \geq 0$

① Min $f(x)$

$g \leq 0$

$\lambda_J \geq 0$

② Max $f(x)$

$g \leq 0$

$\lambda \leq 0$

③ Min $f(x)$

$g \geq 0$

$\lambda_J \leq 0$

max $f(x)$

$g \geq 0$

$\lambda \geq 0$

Q. Use KT condition to minimize $f(x,y,z) = x^2 + y^2 + z^2 + 20x + 10y$

S.T. (i) $x \geq 40$

(ii) $x + y \geq 80$

(iii) $x + y + z \geq 120$

Sol^n:- $L = x^2+y^2+z^2+20x+10y+\lambda_1(x-40)+\lambda_2(x+y-80)$
$+\lambda_3(x+y+z-120)$

I
$$\frac{\partial L}{\partial x} = 2x+20+\lambda_1+\lambda_2+\lambda_3=0 \quad \text{--- (1)}$$
$$\frac{\partial L}{\partial y} = 2y+10+\lambda_2+\lambda_3=0 \quad \text{--- (2)}$$
$$\frac{\partial L}{\partial z} = 2z+\lambda_3=0 \quad \text{--- (3)}$$

II
$$\lambda_1(x-40)=0 \quad \text{--- (4)}$$
$$\lambda_2(x+y-80)=0 \quad \text{--- (5)}$$
$$\lambda_3(x+y+z-120)=0 \quad \text{--- (6)}$$

III
$$x-40\geq 0 \quad \text{--- (7)}$$
$$x+y-80\geq 0 \quad \text{--- (8)}$$
$$x+y+z-120\geq 0 \quad \text{--- (9)}$$

IV
$$\lambda_j \leq 0$$
$$\lambda_1\ \lambda_2\ \lambda_3 \leq 0 \quad \text{--- (10)}$$

From eq^n - (4), (5) & (6)

~~A manufacturing firm producing~~

if $\lambda_1 \neq 0$, $\lambda_3 \neq 0$, $\lambda_2 \neq 0$

$x-40=0$, $x+y+z=120$

$x=40$, $x+y=80$

$y=40$, $z=40$

from eq^n (1), (2) & (3), $\boxed{\lambda_3=-80}$

$\lambda_2 = -2(40)-10-\lambda_3$

$\lambda_2 = -80-10+80=-10$ $\boxed{\lambda_2=-10}$

$\lambda_1 = -80-20-(-10)-(-80)$

$\boxed{\lambda_1=-10}$

Thus the conditions $d_i \leq 0 \quad i=1,2,3$ are also satisfied here. So the solution is

$$x_0 = y = z = 40$$

$$Z_{min} = 3 \times 40^2 + (20+10) \times 40 = 6000 \text{ Ans}$$

Q. A manufacturing firm producing small refrigerator has entered into a contact to ~~sma~~ supply 50 refrigerators at the end of the first month, 50 at the end of the second month, and 50 at the end of the third. The cost of producing x refrigerators in any month is given by $\$ (x^2 + 1000)$. The firm can produce more refrigerator in any month and carry them to a subsequent month. However it costs \$ 20 per unit for any refrigerator carried over from one month to next. Assuming that there is no initial inventory, determine no. of refrigerators to be produced in each month to minimize the total cost.

Ans - Let x_1, x_2, x_3 are the no. of refrigertor per month respectively

Total cost = production cost + holding cost

$$\min f = x_1^2 + 1000 + x_2^2 + 1000 + x_3^2 + 1000 + 20(x_1 - 50) + 20(x_1 + x_2 - 100)$$

$$f_{min} = x_1^2 + x_2^2 + x_3^2 + 40x_1 + 20x_2$$

$$x_1 \geq 50$$

$$x_1 + x_2 \geq 100$$

$x_1 + x_2 + x_3 \geq 150$

$$L = x_1^2 + x_2^2 + x_3^2 + 40x_1 + 20x_2 + \lambda_1(x_1 - 50) + \lambda_2(x_1 + x_2 - 100) + \lambda_3(x_1 + x_2 + x_3 - 150)$$

(I) $\dfrac{\partial L}{\partial x_1} = 2x_1 + 40 + \lambda_1 + \lambda_2 + \lambda_3 = 0$ — (1)

$\dfrac{\partial L}{\partial x_2} = 2x_2 + 20 + \lambda_2 + \lambda_3 = 0$ — (2)

$\dfrac{\partial L}{\partial x_3} = 2x_3 + \lambda_3 = 0$ — (3)

(II) $\lambda_i g_j = 0$

$\lambda_1(x_1 - 50) = 0$ — (4)

$\lambda_2(x_1 + x_2 - 100) = 0$ — (5)

$\lambda_3(x_1 + x_2 + x_3 - 150) = 0$ — (6)

(III) $g_i \geq 0$

$x_1 - 50 \geq 0$ — (7)

$x_1 + x_2 - 100 \geq 0$ — (8)

$x_1 + x_2 + x_3 - 150 \geq 0$ — (9)

(IV) $\lambda_i \leq 0$

$\lambda_1, \lambda_2, \lambda_3 \leq 0$ — (10)

(1) $\rightarrow \quad x_1 = -20 - \frac{\lambda_1}{2} - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}$

(2) $\rightarrow \quad x_2 = -10 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}$

(3) $\rightarrow \quad x_3 = -\frac{\lambda_3}{2}$

(4) $\rightarrow \quad \lambda_1 (x_1 - 50) = 0$

case I

$\lambda_1 = 0, \quad x_1 - 50 \neq 0$

(3) $\quad$ $\left. \begin{aligned} x_1 &= -20 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2} \\ x_2 &= -10 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2} \end{aligned} \right] \text{— (11)}$

(5) $\rightarrow \quad \lambda_2 (x_1 + x_2 - 100) = 0$

$$\lambda_2 \left[\left(-20 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}\right) + \left(-10 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}\right) + 100 \right] = 0$$

$$\lambda_2 (-130 - \lambda_2 - \lambda_3) = 0 \quad \text{— (12)}$$

(6) $\rightarrow \quad \lambda_3 (x_1 + x_2 + x_3 - 150) = 0$

$$\lambda_3 \left[\left(-20 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}\right) + \left(-10 - \frac{\lambda_2}{2} - \frac{\lambda_3}{2}\right) - \frac{\lambda_3}{2} - 150 \right] = 0$$

eq^n (12) & (13) $\quad \lambda_3 \left[-180 - \lambda_2 - \frac{3\lambda_3}{2} \right] = 0 \quad \text{— (13)}$

case-A

when $\lambda_2 = 0$

$\lambda_3 \neq 0, \quad -180 - \lambda_2 - \frac{3\lambda_3}{2} = 0$

$$\lambda_3 = \frac{-180 \times 2}{3} \Rightarrow \lambda_3 = -120$$

from eq^n - ①, ② & ③

$x_3 = 60$, $x_2 = 50$, $x_1 = 40$

in this case at eq^n - ⑦

$$x_1 - 50 \geq 0$$
$$40 - 50 \geq 0$$
$$-10 \geq 0$$

That is false so this case can not be applied.

Case - B

So now $\lambda_2 = 0$ then $\lambda_3 = 0$, $-180 - \lambda_2 - \frac{3\lambda_3}{2} \neq 0$

then in eq^n - ①, ② & ③

$x_1 = -20$, $x_2 = -10$, $x_3 = 0$

This case does not satisfies the eq^n condition.

Case - C

when $\lambda_2 \neq 0$, $-130 = \lambda_2 + \lambda_3$ — ⑭

$\lambda_3 \neq 0$, $-180 - \lambda_2 - \frac{3\lambda_3}{2} = 0$ — ⑮

$$-180 = \cancel{\lambda_2} + \frac{3\lambda_3}{2}$$
$$-130 = \cancel{\lambda_2} + \lambda_3$$
$$\underline{+ \quad\quad - \quad\quad -}$$
$$-50 = \frac{1}{2}\lambda_3 \Rightarrow \boxed{\lambda_3 = -100}, \boxed{\lambda_2 = -30}$$

now in eq^n - ①, ② & ③

$x_3 = 50$, $x_2 = -10 + 15 + 50 = 55$, $x_1 = 45$

But in condition $x_1 - 50 \geq 0$

$45 - 50 \geq 0 \Rightarrow -5 \geq 0$

This case does not satisfy the condition.

Case-D when $\lambda_2 \neq 0$, $-130 = \lambda_2 + \lambda_3$

$\lambda_3 = 0$, $-180 - \lambda_2 - \frac{3\lambda_3}{2} \neq 0$

$\lambda_3 = 0, \quad \lambda_2 = -130$

in eqⁿ - ①, ② & ③

$x_3 = 0, \quad x_2 = 55, \quad x_1 = 45$

$45 - 50 \geq 0 \Rightarrow -5 \geq 0$

This case ~~does~~ also not satisfy the condition.

Now

Case - II $\quad \lambda_1 \neq 0, \quad x_1 - 50 = 0$

$x_1 = 50$

eqⁿ ③ $\quad \lambda_3 = -2x_3$

eqⁿ - ② $\quad 2x_2 + 20 + \lambda_2 - 2x_3 = 0$

$\lambda_2 = 2x_3 - 2x_2 - 20$

eqⁿ - ① $\quad 2x_1 + 40 + \lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 = 2x_2 - 2x_3 - 40 + 2x_3 - 2x_1 + 20$

$\lambda_1 = 2x_2 - 40 - 100 + 20$

$\lambda_1 = -120 + 2x_2$ — ⑯

In eqⁿ - ⑤ & ⑥

$(2x_3 - 2x_2 - 20)(x_1 + x_2 - 100) = 0$ — ⑰

$(-2x_3)(x_1 + x_2 + x_3 - 150) = 0$ — ⑱

Solⁿ of eqⁿ - ⑰ & ⑱ $\Rightarrow$

Case-A $\quad 2x_3 - 2x_2 - 20 = 0, \quad -2x_3 = 0$

then $\quad x_2 = -10, \; x_3 = 0, \; x_1 = 50$

But

eqⁿ ⑧ & ⑨ are not satisfied by this

$-10 - 100 \geq 0 \quad -110 \geq 0$

$40 - 150 \geq 0 \quad -110 \geq 0$] false

Case-B $\quad 2x_3 - 2x_2 - 20 = 0, \quad x_1 + x_2 + x_3 = 150$

$x_2 + x_3 = 100$

$-x_2 + x_3 = 10$

$2x_3 = 110$

$x_3 = 55$, $x_2 = 65$, $x_1 = 50$

in eqn - 8, 9

eqn - 8 $\quad 85 - 100 \geq 0$

$-15 \geq 0$ ✗ not satisfied

eqn - 9 $\quad 50 + 65 + 55 - 150 \geq 0$

$170 - 150 \geq 0$

$20 \geq 0$ ✓

Case - C $\quad x_1 + x_2 = 100$, $x_3 = 0$

then $x_1 = 50$, $x_2 = 50$, $x_3 = 0$

eqn - 8 $\quad 100 - 100 \geq 0$ ✓

eqn - 9 $\quad 50 + 50 - 150 \geq 0$

$-50 \geq 0$ ✗ not satisfied

Case - D

$x_1 + x_2 = 100$, $x_1 + x_2 + x_3 = 150$

$x_1 = 50$, $x_2 = 50$, $x_3 = 50$

eqn - 8 $\quad 100 - 100 \geq 0$ ✓

eqn - 9 $\quad 150 - 150 \geq 0$ ✓

all the eqn are satisfied by these values

Now eqn - 16

$\lambda_2 = 2(50) - 2(50) - 20$

$\boxed{\lambda_2 = -20}$

$\lambda_1 = -120 + 2(50) = -20$

$\boxed{\lambda_1 = -20}$

$\lambda_3 = -2x_3 = -2(50)$

$\boxed{\lambda_3 = -100}$

So this solution is optimum solution.

Thus $x_1 = 50$, $x_2 = 50$, $x_3 = 50$

$f_{min} = (50)^2 \times 3 + 40 \times 50 + 20 \times 50$

$= 2500 \times 3 + 2000 + 1000$

$= 7500 + 3000 = 10500$ Ans

Q. $\min f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

S.T.
$$g_1 = 2x_1 + x_3 - 5 \le 0$$
$$g_2 = x_1 + x_3 - 2 \le 0$$
$$g_3 = 1 - x_1 \le 0$$
$$g_4 = 2 - x_2 \le 0$$
$$g_5 = -x_3 \le 0$$

<u>A</u> -

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda_1 (2x_1 + x_3 - 5) + \lambda_2 (x_1 + x_3 - 2) + \lambda_3 (1 - x_1) + \lambda_4 (2 - x_2) + \lambda_5 (-x_3)$$

I
$$\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda_1 + \lambda_2 - \lambda_3 = 0 \quad \text{——— (1)}$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_4 = 0 \quad \text{——— (2)}$$
$$\frac{\partial L}{\partial x_3} = 2x_3 + \lambda_1 + \lambda_2 - \lambda_5 = 0 \quad \text{——— (3)}$$

II
$$\lambda_1 (2x_1 + x_3 - 5) = 0 \quad \text{——— (4)}$$
$$\lambda_2 (x_1 + x_3 - 2) = 0 \quad \text{——— (5)}$$
$$\lambda_3 (1 - x_1) = 0 \quad \text{——— (6)}$$
$$\lambda_4 (2 - x_2) = 0 \quad \text{——— (7)}$$
$$\lambda_5 (-x_3) = 0 \quad \text{——— (8)}$$

III
$$2x_1 + x_3 - 5 \le 0 \quad \text{——— (9)}$$
$$x_1 + x_3 - 2 \le 0 \quad \text{——— (10)}$$
$$1 - x_1 \le 0 \quad \text{——— (11)}$$
$$2 - x_2 \le 0 \quad \text{——— (12)}$$
$$-x_3 \le 0 \quad \text{——— (13)}$$

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IV $\quad \lambda_5 \geq 0$

$\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \geq 0$ —— 14

~~If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 \neq 0$~~

~~$2x_1 + x_3 - 5 = 0$~~ ~~If $\lambda_1 \neq 0$~~, $2x_1 + x_3 = 5$

~~$x_1 + x_3 - 2 = 0$~~

$\boxed{x_1 = 3}$

Case - I

$\lambda_2 \neq 0$, $x_1 + x_3 = 2$

$\boxed{\lambda_3 = 0}$, $1 \neq x_1$

$\boxed{\lambda_4 \neq 0}$ $\quad \boxed{2 = x_2}$

$\boxed{\lambda_5 = 0}$, $x_3 \neq 0$

$2x_1 + x_3 = 5$

$x_1 + x_3 = 2$

$\boxed{x_1 = 3}$ $\quad \boxed{x_3 = -1}$

Case I $\quad \lambda_1 = 0$, $2x_1 + x_3 - 5 \neq 0$

~~$\lambda_3 = \lambda_5 = 0$, $x_1 = 3$; $x_3 = -1$, $x_2 = 2$~~

1 →

$$x_1 = \frac{\lambda_3}{2} - \frac{\lambda_2}{2} - \lambda_1$$

2 → $x_2 = \dfrac{\lambda_4}{2}$ —— 15

3 → $x_3 = \dfrac{\lambda_5 - \lambda_2 - \lambda_1}{2}$

$$\lambda_2 \left(\frac{\lambda_3}{2} - \frac{\lambda_2}{2} - \lambda_1 + \frac{\lambda_5}{2} - \frac{\lambda_2}{2} - \frac{\lambda_1}{2} - 2 \right) = 0$$

$$\lambda_2 \left(\frac{-3\lambda_1}{2} - \lambda_2 - \frac{\lambda_3}{2} + \frac{\lambda_5}{2} - 2 \right)$$

$$\lambda_3 \left(1 + \lambda_1 + \frac{\lambda_2}{2} - \frac{\lambda_3}{2} \right)$$ —— 16

$$\lambda_4 \left(2 - \frac{\lambda_4}{2} \right)$$

$$\lambda_5 \left(\frac{\lambda_1 + \lambda_2 - \lambda_5}{2} \right)$$

If $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, $\lambda_4 \neq 0$, $\lambda_5 = 0$

$x_3 = 0$, $x_2 = 2$, $x_1 = 1$, $x_1 + x_3 \neq 2$,

$2x_1 + x_3 \neq 5$

According to condition

All the eq are satisfied by these values.

eq^n - 9 $\quad 2x_1 + x_3 - 5 \leq 0$

$2 - 5 \leq 0$

$-3 \leq 0$ ✓

eq^n - 10 $\quad x_1 + x_3 - 2 \leq 0$

$1 - 2 \leq 0 \Rightarrow -1 \leq 0$ ✓

eq^n - 11 $\quad 1 - x_1 \leq 0$

$0 \leq 0$ ✓

eq^n - 12 $\quad 2 - x_2 \leq 0$

$0 \leq 0$ ✓

eq^n - 13

$0 \leq 0$ ✓

$$2 - \frac{\lambda_4}{2} = 0$$

$$\boxed{\lambda_4 = 4} \qquad \boxed{\lambda_3 = 2} \qquad \boxed{\lambda_5 = 0}$$

So $x_1 = 1$, $x_2 = 2$, $x_3 = 0$

$f_{min} = (1)^2 + (2)^2 + (0)^2$

$f_{min} = 1 + 4$

$$\boxed{f_{min} = 5}$$

Unit - 5

LINEAR PROGRAMMING

$$Z = c_1 x_1 + c_2 x_2 + \cdots c_n x_n$$

$$a_{11}x_1 + a_{12}x_2 - \cdots a_{1m}x_m \leq b_1 \atop \geq$$

$$a_{n1}x_1 + a_{n2}x_2 - \cdots a_{mn}x_m \leq \atop = \atop \geq \; b_m$$

$$\left.\begin{array}{l} Z = CX \\ AX \leq, \geq, = \; b \\ X \geq 0 \end{array}\right]$$

Basic variable & solution →

★ Simplex method → (function should be maximum.)

$$\text{Min } Z = 2x_1 + 5x_2$$

$$\text{Max } Z = -2x_1 - 5x_2$$

[convert ~~on~~ into max. by multiple - sign.]

$$2x_2 \leq -3$$

$$-2x_2 \geq 3$$

Q. ★ Solve the ~~question~~ following LPP ~~editing~~ by simplex method

$$\max Z = 4x_1 + 5x_2$$

S.T.

$$x_1 + x_2 \leq 3$$

$$3x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Sol^n:-

$$\text{Max } Z = 4x_1 + 5x_2$$

$$x_1 + x_2 + \underbrace{x_3}_{\text{slack variable}} = 3$$

$$3x_1 + 4x_2 + x_4 = 10$$

$$x_1 x_2 x_3 x_4 \geq 0$$

$1\times0+3\ldots$ $R_1 \to R_1 - R_2$

$\frac{10}{4} = \frac{5}{2}$

$$\max Z = 4x_1 + 5x_2 + 0\cdot x_3 + 0\cdot x_4$$

		C_j	4	5	0	0	Min Ratio
C_B	B.V.	X_B	y_1	y_2	y_3	y_4	
0	x_3	3	1	1	1	0	$3/1 = 3$
0	x_4	10	3	$\boxed{4}$ key element	0	1	$10/4 = 2.5$
$Z_j - C_j$			-4	-5 entering/incoming element	0	0	outgoing
0	x_3	$1/2$	$1/4$	0	1	$-1/4$	2
5	x_2	$5/2$	$3/4$	1	0	$1/4$	$10/3 = 3.3$
$Z_j - C_j$			$-1/4$ entering/incoming element	0	0	$5/4$ outgoing	
4	x_1	2	1	0	4	-1	
5	x_2	1	0	1	-3	1	
$Z_j - C_j$			0	0	1	1	

-4

$R_2 \to R_2 - \frac{3}{4}R_1$

$x_1 = 2$

$x_2 = 1$

$\max Z = 4x_1 + 5x_2$

$= 8 + 5 = 13$

$\boxed{\max Z = 13}$

Q. Solve the ~~entity~~ LPP by simplex method

$\text{Min } Z = x_1 - 3x_2 + 2x_3$

s.T. $3x_1 - x_2 + 3x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \geq 0$

Min $z = -\text{Max}(-z')$

Ay – $\text{Max } z' = -x_1 + 3x_2 - 2x_3$

$3x_1 - x_2 + 3x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

$x_1 x_2 x_3 x_4 x_5 x_6 \geq 0$

$\text{Max } z' = -x_1 + 3x_2 - 2x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6$

		c_j	-1	3	-2	0	0	0	
C_B	B.V.	x_B	y_1	y_2	y_3	y_4	y_5	y_6	Min Ration $\frac{x_B}{y_j}$
0	x_4	7	3	-1	3	1	0	0	—
0	x_5	12	-2	$\boxed{4}$	0	0	1	0	$12/4 = 3$
0	x_6	10	-4	3	8	0	0	1	$10/3 = 3.3$
$z_j - c_j$			1	-3 ↑	2	0	0 ↓	0	
0	x_4	10	$\boxed{5/2}$	0	3	1	1/4	0	$10/\frac{5}{2} = 4$
3	x_2	3	-1/2	1	0	0	1/4	0	—
0	x_6	1	-5/2	0	8	0	-3/4	1	—
$z_j - c_j$			-1/2 ↑	0	2	0 ↓	3/4	0	
-1	x_1	4	1	0	6/5	2/5	1/10	0	
3	x_2	5	0	1	3/5	1/5	3/10	0	
0	x_6	11	0	0	11	1	-1/2	1	
$z_j - c_j$			0	0	13/5	1/5	4/5	0	

$x_1 = 4$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$

$x_2 = 5$

$x_6 = 11$

$$\text{Max } z' = -x_1 + 3x_2 - 2x_3 + 0\cdot x_4 + 0\cdot x_5 + 0x_6$$

$$\text{Max } z' = -4 + 3\times 5 - 2\times 0 + 0\cdot 0 + 0\cdot 0 + 0\cdot 11$$

$$\text{Max } z' = 11$$

$$\text{Min } z = -\text{Max}(z)$$

$$\boxed{\text{Min } z = 11}$$

Q. Solve the LPP by Simplex method

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

S.T.

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$x_1 x_2 x_3 \geq 0$$

Ans -

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

$$2x_1 + 3x_2 + x_4 = 8$$

$$2x_2 + 5x_3 + x_5 = 10$$

$$3x_1 + 2x_2 + 4x_3 + x_6 = 15$$

$$x_1 x_2 x_3 x_4 x_5 x_6 \geq 0$$

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0\cdot x_4 + 0\cdot x_5 + 0\cdot x_6$$

	C_j		3	5	4	0	0	0	
C_B	B.V.	X_B	y_1	y_2	y_3	y_4	y_5	y_6	Min Ratio $\frac{X_B}{y_j}$
0	x_4	8	2	$\boxed{3}$	0	1	0	0	$8/3 = 2.67$
0	x_5	10	0	2	5	0	1	0	$10/2 = 5$
0	x_6	15	3	2	4	0	0	1	$15/2 = 7.5$
	$Z_j - C_j$		-3	$-5\uparrow$	-4	$0\downarrow$	0	0	
5	x_2	$8/3$	$2/3$	1	0	$1/3$	0	0	—
0	x_5	$14/3$	$-4/3$	0	$\boxed{5}$	$-2/3$	1	0	$14/15 = 0.9$
0	x_6	$29/3$	$5/3$	0	4	$-2/3$	0	1	$29/12 = 2.41$
	$Z_j - C_j$		$1/3$	0	$-4\uparrow$	$5/3$	$0\downarrow$	0	
5	x_2	$8/3$	$\boxed{2/3}$	1	0	$1/3$	0	0	$8/2 = 4$
4	x_3	$14/15$	$-4/15$	0	1	$-2/15$	$1/5$	0	—
0	x_6	$89/15$	$\boxed{41/15}$	0	0	$-2/15$	0	1	$89/41 = 2.1$
			$-17/15\uparrow$	0	0	$17/15$	$4/5$	$0\downarrow$	
5	x_2	$50/41$	0	1	0	$15/41$	$8/41$	$10/41$	
4	x_3	$62/41$	0	0	1	$-6/41$	$5/41$	$4/41$	
3	x_1	$89/41$	1	0	0	$\frac{-2}{41}$	0	$15/41$	
	$Z_j - C_j$		0	0	0	$\frac{60}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_3 \to R_3 - 4R_2$

$R_1 \to R_1 - \frac{2}{3}R_3$

$R_2 \to R_2 + \frac{4}{15}R_3$

$x_2 = \frac{50}{41}, \quad x_3 = \frac{62}{41}, \quad x_1 = \frac{89}{41}$

$Max\ Z = \frac{3\times 50}{41} + \cancel{5\times 0}$

$$Max\ Z = \frac{3\times 89}{41} + \frac{5\times 50}{41} + \frac{4\times 62}{41} = \frac{267+250+248}{41}$$

$$\boxed{Max\ Z = \frac{765}{41}}$$

Er Sahil Ka Gyan

Solution the problem with artificial variable

⇒

(i) 2 phase method :—

Q. $\text{Min } z = x_1 + x_2$

S.T. $2x_1 + 4x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Ans – $\text{Max } Z = -x_1 - x_2$

$2x_1 + 4x_2 - x_3 + x_5 = 4$

$x_1 + 7x_2 - x_4 + x_6 = 7$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

phase -1

$\text{Max } Z = -0 \cdot x_1 - 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 - x_5 - x_6$

		C_j	0	0	0	0	-1	-1	
C_B	B.V.	X_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio $X_B/y_{i}, y_i>0$
-1	x_5	4	2	4	-1	0	1	0	$4/4 = 1$
-1	x_6	7	1	$\boxed{7}$	0	-1	0	1	$7/7 = 1$
$Z_j - C_j$			-3	-11 ↑	1	1	0	0 ↓	
-1	x_5	3	$\boxed{13/7}$	0	-1	1/7	1		$21/13 = 1.6$
0	x_2	1	1/7	1	0	-1/7	0		7
$Z_j - C_j$			-13/7 ↑	0	1	-1/7	0 ↓		
0	x_1	21/13	1	0	-7/13	1/13			
0	x_2	10/13	0	1	1/13	-2/13			
$Z_j - C_j$			0	0	0	0			

phase - 2

C_B	B.V.	C_j / X_B	-1 / y_1	-1 / y_2	0 / y_3	0 / y_4
-1	x_1	21/13	1	0	-7/13	1/13
-1	x_2	10/13	0	1	1/13	-2/13
$Z_j - C_j$			0	0	6/13	1/13

$x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$

$\text{Max } Z = -x_1 - x_2 + 0 \cdot x_3 + 0 \cdot x_4$

$\text{Max } Z = \frac{-21-10}{13} = \frac{-31}{13}$

$\text{Min } Z = -\text{Max } Z$

$\boxed{\text{Min } Z = \frac{31}{13}}$

Q. $\text{Max } Z = -x_1 - x_2$

S.T. $2x_1 - 3x_2 \geq 4$

$-2x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$

A- $\text{Max } Z = -x_1 - x_2$

$2x_1 - 3x_2 - x_3 + x_5 = 4$

$-2x_1 - x_2 - x_4 + x_6 = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

phase 1

$\text{Max } Z = -0 \cdot x_1 - 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 - x_5 - x_6$

		C_j	0	0	0	0	-1	-1	
C_B	B.V	X_B	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
-1	x_5	4	2	-3	-1	0	1	0	
-1	x_6	0	-2	-1	0	-1	0	1	
	$Z_j - C_j$		0	4	1	1	0	0	

phase-2

		C_j	-1	-1	0	0	0	0
C_B	B.V	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_5	4	2	-3	-1	0	1	0
0	x_6	0	-2	-1	0	-1	0	1
	$Z_j - C_j$		1	1	0	0	0	0

Hence, There is no physical solution coz
$x_5 = 4$, $x_6 = 0$ artificial variable can not be removed.

$$Max\ Z = 0$$

Ans.

* Duality :—

$$\text{Max } Z_p = CX$$
$$AX \le b$$
$$X \geq 0$$

$$\text{Min } Z_D = b^T w$$
$$A^T w \geq C^T$$
$$w \geq 0$$

$$\text{Max } Z_p = CX$$
$$AX \le b$$
$$X \geq 0$$

If primar has unbounded then dual will be not feasible.

If $Z_p \rightarrow$ no , $Z_D \rightarrow$ no or unbounded

$AX = b$ is unsymmetric condition

if $x_1 + x_2 = 4$ then convert

$x_1 + x_2 \le 4$

$x_1 + x_2 \geq 4$

Q. Write the dual of the problem.

$$\text{Max } Z = 2x_1 + 4x_2$$

S.T. $2x_1 + 3x_2 \le 48$

$x_1 + 3x_2 \le 42$

$x_1 + x_2 \le 21$

$x_1, x_2 \geq 0$

Ans — $\text{Min } Z_D = 48w_1 + 42w_2 + 21w_3$

$2w_1 + w_2 + w_3 \geq 2$

$3w_1 + 3w_2 + w_3 \geq 4$

$w_1, w_2, w_3 \geq 0$

Q. Min $Z = 3x_1 + x_2$

S.T. $2x_1 + 3x_2 \geq 2$

$x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

Ay-

$$\text{Max } Z = 2w_1 + w_2$$

$$2w_1 + w_2 \leq 3$$

$$3w_1 + w_2 \leq 1 \qquad w_1, w_2 \geq 0$$

Q. Min $Z_p = 2x_2 + 5x_3$

S.T. $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2 \geq 0$

Ay - ~~Max e Zp=~~

$$\text{Min } Z_p = 2x_2 + 5x_3$$

$$x_1 + x_2 \geq 2$$

$$-2x_1 - x_2 - 6x_3 \geq -6$$

$$x_1 - x_2 + 3x_3 \geq 4$$

$$-x_1 + x_2 - 3x_3 \geq -4$$

$$x_1, x_2, x_3 \geq 0$$

- To convert into Dual :—

$$\text{Max } Z_D = 2w_1 - 6w_2 + 4w_3 - 4w_4$$

$$w_1 - 2w_2 + w_3 - w_4 \leq 0$$

$$w_1 - w_2 - w_3 + w_4 \leq 2$$

$$-6w_2 + \underbrace{3w_3 - 3w_4}_{w_3'} \leq 5$$

or

~~Min Zp~~ Max $Z_P = -2x_2 - 5x_3$

$-x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 \leq 4$

$-x_1 + x_2 - 3x_3 \leq -4$

Min $Z_D = -2w_1 + 6w_2 + 4w_3 - 4w_4$

$-w_1 + 2w_2 + w_3 - w_4 \geq 0$

$-w_1 + w_2 - w_3 + w_4 \geq -2$

$6w_2 + 3w_3 - 3w_4 \geq -5$

$w_1\ w_2\ w_3 w_4 \geq 0$

$-w_1 + 2w_2 + (w_3 - w_4) \geq 0$

$-w_1 + w_2 - (w_3 - w_4) \geq -2$

$6w_2 + 3(w_3 - w_4) \geq -5$

let $w' = w_3 - w_4$

$-w_1 + 2w_2 + w_3' \geq 0$

$-w_1 + w_2 - w_3' \geq -2$

$6w_2 + 3w_3' \geq -5$

$w_1 w_2 \geq 0$

w_3' unrestricted

Q. Min $Z = x_1 + x_2 + x_3$

s.t. $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_1 - x_3 \geq 4$

$x_1\ x_2 \geq 0$ and x_3 is unrestricted

By -

$$\text{Min } z = x_1 + x_2 + x_3$$

$$x_1 - 3x_2 + 4x_3 \leq 5$$

$$x_1 - 3x_2 + 4x_3 \geq 5$$

$$x_1 - 2x_2 \leq 3$$

$$2x_1 - x_3 \geq 4$$

$$\text{Min } z = x_1 + x_2 + x_3$$

$$-x_1 + 3x_2 - 4x_3 \geq -5$$

$$x_1 - 3x_2 + 4x_3 \geq 5$$

$$-x_1 + 2x_2 \geq -3$$

$$2x_1 - x_3 \geq 4$$

$x_1, x_2 \geq 0$ $\quad$ x_3 unrestricted

$$x_3 = x_3' - x_3''$$

$$\text{Min } z = x_1 + x_2 + x_3' - x_3''$$

$$-x_1 + 3x_2 - 4(x_3' - x_3'') \geq -5$$

$$x_1 - 3x_2 + 4(x_3' - x_3'') \geq 5$$

$$-x_1 + 2x_2 \geq -3$$

$$2x_1 - (x_3' - x_3'') \geq 4$$

$$x_1, x_2, x_3', x_3'' \geq 0$$

On converting into Dual:—

$$\text{Max } Z_D = -5w_1 + 5w_2 - 3w_3 + 4w_4$$

$$-w_1 + w_2 - w_3 + 2w_4 \leq 1$$

$$3w_1 - 3w_2 + 2w_3 + 0 \cdot w_4 \leq 1$$

$$-4w_1 + 4w_2 + 0 \cdot w_3 - w_4 \leq 1$$

$$4w_1 - 4w_2 + 0 \cdot w_3 + w_4 \leq -1$$

$$w_1, w_2, w_3, w_4 \geq 0$$

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Q. Use duality to solve the following entity:—

$$\text{Min } z = 2x_1 + 9x_2 + x_3$$

S.to. $x_1 + 4x_2 + 2x_3 \geq 5$

$3x_1 + x_2 + 2x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

dy -

$$\text{Max } z = 5w_1 + 4w_2$$

$w_1 + 3w_2 \leq 2$

$4w_1 + w_2 \leq 9$

$2w_1 + 2w_2 \leq 1$

$w_1, w_2, w_3 \geq 0$

$$\text{Max } z = 5w_1 + 4w_2 + 0 \cdot w_3 + 0 \cdot w_4 + 0 \cdot w_5$$

S.to. $w_1 + 3w_2 + w_3 = 2$

$4w_1 + w_2 + w_4 = 9$

$2w_1 + 2w_2 + w_5 = 1$

		C_j	5	4	0	0	0	
C_B	B.V.	X_B	y_1	y_2	y_3	y_4	y_5	Min ratio
0	w_3	2	1	3	1	0	0	2
0	w_4	9	4	1	0	1	0	$9/4 = 2.25$
0	w_5	1	$\boxed{2}$	2	0	0	1	$1/2 = 0.5$
	$Z_j - C_j$		$-5\uparrow$	-4	0	0	$0\downarrow$	
0	w_3	$3/2$	0	2	1	0	$-1/2$	
0	w_4	7	0	-3	0	1	-2	
5	w_1	$1/2$	1	1	0	0	$1/2$	
	$Z_j - C_j$		0	1	0	0	$5/2$	

$w_3 = 3/2$ $\qquad w_2 = w_5 = 0$

$w_4 = 7$

$w_1 = 1/2$

$$\text{Max } Z = 5w_1 + 4w_2$$

$$\text{Max } z = 5 \times \frac{1}{2} + 4 \times 0 = 5/2$$

Q. Solve the following and its dual by simplex method and then compare their solution.

Ans — $\text{Min } z = 3x_1 + x_2$

S.To.

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

$\text{Max } z = -3x_1 - x_2$

S.To.

$x_1 + x_2 - x_3 + x_5 = 1$

$2x_1 + 3x_2 - x_4 + x_6 = 2$

$x_i \geq 0 \quad i = 1 \text{ to } 6$

Using Big-M method

$$\text{Max } z = -3x_1 - x_2 + 0 \cdot x_3 + 0 \cdot x_4 - Mx_5 - Mx_6$$

		c_j	-3	-1	0	0	-M	-M	
C_B	B.V.	X_B	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
-M	x_5	1	1	1	-1	0	1	0	1
-M	x_6	2	2	$\boxed{3}$	0	-1	0	1	$2/3 = 0.6$
$z_j - c_j$			$-3M+3$	$-4M+1 \uparrow$	M	M	0	$0 \downarrow$	
-M	x_5	$1/3$	$1/3$	0	-1	$\boxed{1/3}$	1		1
-1	x_2	$2/3$	$2/3$	1	0	$-1/3$	0		—
$z_j - c_j$			$-\frac{M}{3} + \frac{7}{3}$	0	M	$-\frac{M}{3} + \frac{1}{3} \uparrow$	$0 \downarrow$		

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0	x_4	1	1	0	-3	1	
-1	x_2	1	1	1	-1	0	
	$z_j - c_j$		2	0	1	0	

$x_4 = 1$

$x_2 = 1$

$\text{Max } Z = -3x_1 - x_2$

$\text{Max } Z = -0 - 1 + 0$

$\boxed{\text{Max } Z = -1}$

$\boxed{\text{Min } Z = 1}$

dual of z $\Rightarrow$

$\text{Max } Z = w_1 + 2w_2$

$w_1 + 2w_2 \leq 3$

$w_1 + 3w_2 \leq 1$

$w_1, w_2 \geq 0$

$\text{Max } Z = w_1 + 2w_2 + 0 \cdot w_3 + 0 \cdot w_4$

$w_1 + 2w_2 + w_3 = 3$

$w_1 + 3w_2 + w_4 = 1$

$w_1, w_2, w_3, w_4 \geq 0$

		c_j	1	2	0	0	
C_B	BV	X_B	y_1	y_2	y_3	y_4	
0	w_3	3	1	2	1	0	$3/2 = 1.5$
0	w_4	1	1	$\boxed{3}$	0	1	$1/3 = 0.3$
	$z_j - c_j$		-1	$-2 \uparrow$	0	$0 \downarrow$	
0	w_3	7/3	1/3	0	1	-2/3	7
2	w_2	1/3	$\boxed{1/3}$	1	0	1/3	1
	$z_j - c_j$		$-1/3 \uparrow$	$0 \downarrow$	0	2/3	

0	w_3	2	0	-1	1	-1
1	w_1	1	1	3	0	1
$\bar{z}_j - c_j$			0	1	0	1

$w_3 = 2$

$w_1 = 1$

$$\text{Max } Z = 1 + 2(0) + 0(2)$$

$$\boxed{\text{Max } Z = 1}$$

⟹ Application of LP :—

Transportation ⟹

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

S.to. $\displaystyle\sum_{j=1}^{n} x_{ij} \le a_i$ → supply

$\displaystyle\sum_{i=1}^{m} x_{ij} \geq b_j$ → demand

$a_{ij} = b_{ij}$ ⟹ balanced transportation (supply = demand)

format of transportation :—

	To	Supply
from	a_1 a_2 a_3	a_1 a_2 ⋮
Demand	b_1 b_2 — — — b_n	$\Sigma a_{ij} = \Sigma b_{ij}$

Q. find the initial ~~physibl~~ feasible solution to the following element :—

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(a) Northwest method

To

	W_1	W_2	W_3	W_4	W_5	Supply		
From F_1	[30] 7	[10] 6	4	5	9	40	10	0
F_2	8	[20] 5	[10] 6	7	8	30	10	0
F_3	6	8	[5] 9	[15] 6	5	20	15	0
F_4	5	7	7	[5] 8	[5] 6	10	5	0
Demand	30	30	15	20	5	100		
	0	20	5	5	0			
		0	0	0				

Cost :- $30\times7 + 10\times6 + 20\times5 + 10\times6 + \cancel{10}\ 5\times9 + 15\times6 + 5\times8 + 5\times6 = \cancel{635}$

$210 + 60 + 100 + 60 + 45 + 60 + 40 + 30 = 635$

(b) lowest cost entry method :—

	W_1	W_2	W_3	W_4	W_5	Supply	
F_1	[5] 7	6	[15] 4	[20] 5	9	40	25 5
F_2	8	[30] 5	6	7	8	30	
F_3	[15] 6	8	9	6	[5] 5	20	15
F_4	[10] 5	7	7	8	6	10	
Demand	30	30	15	20	5	100	
	20						
	5						

$5\times7 + 30\times5 + 15\times6 + 10\times5 + 20\times5 + 5\times5 + 15\times4$

$35 + 150 + 90 + 50 + 100 + 25 + 60 = 510$

(c) VAM ⇒ —

	W_1	W_2	W_3	W_4	W_5	Supply		
F_1	7	6	[15] 4	5	9	40	(1)	25
F_2	8	5	6	7	8	30	(1)	
F_3	6	8	9	6	5	20	(1)	
F_4	5	7	7	8	6	10	(1)	
Demand	30	30	15	20	5	100		
	(1)	(1)	(2)	(1)	(1)			
			0					

	W_1	W_2	W_4	W_5	Supply	
F_1	7	6	5	9	25	(1)
~~F_2~~	~~8~~	~~$\boxed{30}$ 5~~	~~7~~	~~8~~	~~30~~	~~(2)~~ 0
F_3	6	8	6	5	20	(1)
F_4	5	7	8	6	10	(1)
Demand	30	30	20	5		
	(1)	(1)	(1)	(1)		

	W_1	W_4	W_5	Supply		
F_1	7	$\boxed{20}$ 5	9	25	(2)	5
F_3	8	6	5	20	(1)	
F_4	5	8	6	10	(1)	
Demand	30	20	5			
	(2)	(1)	(1)			
		0				

	W_1	W_5	Supply		
~~F_1~~	~~$\boxed{5}$ 7~~	~~9~~	~~5~~	~~(2)~~	0
F_3	6	5	20	(1)	
F_4	5	6	10	(1)	
Demand	30	5			
	(1)	(1)			
	25				

	W_1	W_5	Supply	
F_3	6	5	20	(1)
~~F_4~~	~~$\boxed{10}$ 5~~	~~6~~	~~10~~	~~(1)~~ 0
	25	5		
	(1)	(1)		
	15			

	W_1	W_5	Supply
F_3	$\boxed{15}$ 6	$\boxed{5}$ 5	20
Demand	25	5	

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~~Cost = 15×4 + 30×5 + 20×5 + 7×5 + 10×5~~

	W1	W2	W3	W4	W5	Supply
F1	5/ 7	6	15/ 4	20/ 5	9	40
F2	8	30/ 5	6	7	8	30
F3	15/ 6	8	9	6	5/ 5	20
F4	10/ 5	7	7	8	6	10
Demand	30	30	15	20	5	100

Cost = 4×15 + 20×5 + 5×7 + 10×5 + 15×6 + 5×8 + 30×5

= 60 + 100 + 35 + 50 + 90 + 40 + 150 + 25

~~270 + 60 = 330~~

$\boxed{\text{Cost} = 510}$ Ans

Optimility Test :—

Modify distribution method (MODI) or U-V method :—

Q. A company has 4 factory F_1 F_2 F_3 & F_4 from which it supplies to 3 warehouse w_1, w_2 & w_3. Determine the optimal transportation from the following data given the factories to warehouses shifting cost, quantities available at each factory and quantities required at each warehouse

	F1	F2	F3	F4	Required	
W1	6	4	1	5	14	(3)
W2	8	9	15/ 2	7	16	(6) 1
W3	4	3	6	2	5	(1)
Available	6	10	15	4	35	
	(2)	(1)	(1)	(3)		
			0			

	F_1	F_2	F_4	Required	
W_1	6	4	5	14	(1)
W_2	8	9	7	1	(1)
W_3	4	3	$\boxed{4}$ 2	5	(1) 1
Available	6	10	4	20	
	(2)	(1)	(3)		
			0		

	F_1	F_2	Required	
W_1	6	$\boxed{10}$ 4	14	(2)
W_2	8	9	1	(1)
W_3	4	3	1	(1)
	6	10		
	(2)	(1)		
		0		

	F_1	Required
W_1	$\boxed{4}$ 6	4
W_2	$\boxed{1}$ 8	1
W_3	$\boxed{1}$ 4	1
Available	6	

		$v_1=0$	v_2	v_3	v_4	
		F_1	F_2	F_3	F_4	Required
u_1	W_1	$\boxed{4}$ 6	$\boxed{10}$ 4	1	5	14
u_2	W_2	$\boxed{1}$ 8	9	$\boxed{15}$ 2	7	16
u_3	W_3	$\boxed{1}$ 4	3	6	$\boxed{4}$ 2	5
	Available	6	10	15	4	

Conditions of optimality test:—

Case-1 if all $d_{ij} > 0$ then solution under test is optimal and unique.

Case-2 If all $d_{ij} \geq 0$ with atleast one $d_{ij} = $ then the solution under test is optimal and an alternative optimal solution exists.

Case-III If at least one $d_{ij} < 0$ then solution is not optimal.

$$\boxed{d_{ij} = C_{ij} - (U_i + V_j)}$$

$C_{rs} = U_r + V_s$ = cost

$C_{11} = u_1 + v_1$

$6 = (u_1 + 0) \Rightarrow u_1 = 6$

$C_{12} = u_1 + v_2$

$4 = 6 + v_2 \Rightarrow v_2 = -2$

$C_{21} = u_2 + v_1$

$8 = u_2$

$C_{23} = u_2 + v_3$

$2 = 8 + v_3 \Rightarrow v_3 = -6$

$C_{31} = u_3 + v_1$

$4 = u_3$

$C_{34} = u_3 + v_4$

$2 = 4 + v_4 \Rightarrow v_4 = -2$

$d_{13} = C_{13} - (u_1 + v_3)$

$1 - (6 - 6) = 1$

$d_{14} = C_{14} - (u_1 + v_4)$

$= 5 - (6 + 2) = \cancel{06}\ 1$

$d_{22} = C_{22} - (u_2 + v_2) = 3$

$d_{24} = C_{24} - (u_2 + v_4) = 1$

$d_{32} = C_{32} - (u_3 + v_2) = 3 - (4 - 2) = 1$

$d_{33} = C_{33} - (u_3 + v_3) = 6 - (4 - 6) = 6 + 2 = 8$

$d_{ij} > 0$ optimal & unique

$= 24 + 40 + 8 + 30 + 4 + 8 = 114$ Ans

Q.

	v_1 F_1	v_2 F_2	v_3 F_3	
$u_1=0$ W_1	$\boxed{5}$ 2	7	4	5
u_2 W_2	3	3	$\boxed{8}$ 1	8
u_3 W_3	5	$\boxed{7}$ 4	7	7
$u_4=0$ W_4	$\boxed{2}$ 1	$\boxed{2}$ 6	$\boxed{10}$ 2	14
	7	9	18	

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$C_{rs} = u_r + v_s$

$C_{11} = u_1 + v_1$

$2 = 0 + v_1$

$\boxed{v_1 = 2}$

$C_{21} = u_2 + v_1$

$3 = u_2 + 2 \Rightarrow \boxed{u_2 = 1}$

$C_{22} = u_2 + v_2$

$3 = 1 + v_2 \Rightarrow \boxed{v_2 = 2}$

$C_{32} = u_3 + v_2$

$4 =$

$1 = u_2 + v_3$

$\cancel{C_{32} = u_3 + v_2}$

$\cancel{1 = u_3 + v_2}$

$4 = u_3 + v_2$

$1 = u_4 + v_1$

$6 = u_4 + v_2$

$2 = u_4 + v_3$

$C_{41} = u_4 + v_1$

$1 = 0 + v_1 = \boxed{v_1 = 1}$

$C_{42} = u_4 + v_2$

$\boxed{6 = v_2}$

$C_{43} = u_4 + v_3$

$\boxed{2 = v_3}$

u_1

u_2

u_3

$u_4 = 0$

$c_{11} = u_1 + v_1$

$2 = u_1 + 1 \Rightarrow \boxed{u_1 = 1}$

$c_{32} = u_3 + v_2$

$4 = u_3 + 6 \Rightarrow \boxed{u_3 = -2}$

$c_{23} = u_2 + v_3$

$1 = u_2 + 2 \Rightarrow \boxed{u_2 = -1}$

$d_{12} = c_{12} - (u_1 + v_2)$

$d_{12} = 7 - (1+6) = 0$

$d_{13} = c_{13} - (u_1 + v_3)$

$d_{13} = 4 - (1+2) = 1$

$d_{21} = c_{21} - (u_2 + v_1)$

$d_{21} = 3 - (-1+1) = 3$

$d_{22} = c_{22} - (u_2 + v_2)$

$d_{22} = 3 - (-1+6) = -2$

$d_{31} = c_{31} - (u_3 + v_1)$

$d_{31} = 5 - (-2+1) = 6$

$d_{33} = c_{33} - (u_3 + v_3)$

$d_{33} = 7 - (-2+2) = 7$

$$\begin{array}{ccc} \theta^{+} & & \bar{8-\theta} \\ 3 & \boxed{8}\,1 & \\ 4 & 7 & \\ \boxed{2}\,6 & \boxed{10}\,2 & 10^{+}+\theta \\ 2-\theta & & \end{array}$$

so now $d_{22} < 0$ then Solution is not optimal.

		v_1 F_1	v_2 F_2	v_3 F_3	
u_1	W_1	[5] 2	7	4	5
2	W_2	3	[2] 3	[6] 1	8
3	W_3	5	[7] 4	7	7
$u_4=0$	W_4	[2] 1	6	[12] 2	14
		7	9	18	

$d_{ij} = c_{ij} - (u_i + v_j)$

$c_{rs} = u_r + v_s$

$c_{41} = u_4 + v_1$

$C_{43} = u_4 + v_3$

$\boxed{2 = v_3}$

$C_{32} = u_3 + v_2$

$4 = u_3 + v_2 \quad \Rightarrow \quad 4 = u_3 + 3 - u_2$

$C_{22} = u_2 + v_2 \qquad\qquad 1 = u_3 - u_2$

$3 = u_2 + v_2$

$C_{11} = u_1 + v_1$

$2 = u_1 + v_1$

$\boxed{u_1 = 1}$

$C_{23} = u_2 + u_3$

$1 = u_2 + u_3$

$1 = -u_2 + u_3$

$2 = 2u_3$

$\boxed{u_3 = 1} \qquad \boxed{u_4 = 0}$

$\boxed{u_2 = 0}$

$4 = 1 + v_2 \quad \rightarrow \quad \boxed{v_2 = 3}$

$u_1 = 1, \; u_2 = 0, \; u_3 = 1, \; v_1 = 1, \; v_2 = 3, \; v_3 = 2$

$d_{12} = C_{12} - (u_1 + v_2)$

$d_{12} = 7 - (1+3) = 3$

$d_{13} = C_{13} - (u_1 + v_3)$

$d_{13} = 4 - (1+2) = 1$

$d_{21} = C_{21} - (u_2 + v_1)$

$d_{21} = 3 - (0+1) = 2$

$d_{31} = C_{31} - (u_3 + v_1)$

$d_{31} = 5 - (1+1) = 3$

$d_{33} = C_{33} - (u_3 + v_3) = 7 - (1+2) = 4$

$d_{42} = c_{42} - (v_2 + u_4)$

$d_{42} = 6 - (3 + 0) = 3$

all positive $d_{ij} > 0$ then solution under test is optimal & unique.

cost = 10 + 6 + 28 + 6 + 24 + 2 = 76

Unbalance Transportation Problem :-

	M_1	M_2	M_3	M_4		Supply
P_1	4	6	8	13	0	50
P_2	13	11	10	8	0	70
P_3	14	4	10	13	0	30
P_4	9	11	13	8	0	50
Demand	25	35	105	20		200

185

	M_1	M_2	M_3	M_4	M_5	Supply	
P_1	4	6	8	13	0	50	(4)
P_2	13	11	10	8	15/0	70	(8) 55
P_3	14	4	10	13	0	30	(4)
P_4	9	11	13	8	0	50	(8)
Demand	25	35	105	20	15		
	(5)	(2)	(2)	(5)	(0)		

0

	M_1	M_2	M_3	M_4		
P_1	4	6	8	13	50	(2)
P_2	13	11	10	8	55	(2)
P_3	14	30/4	10	13	30	(6) 0
P_4	9	11	13	8	50	(1)
	25	35	105	20		
	(5)	(2)	(2)	(5)		
		5				

	M_1	M_2	M_3	M_4			
P_1	4	6	8	13	50	(2)	
P_2	13	11	10	20] 8	55	(2)	35
P_4	9	11	13	8	50	(1)	
	25	5	105	20			
	(5)	(5)	2	(8)			
				0			

	M_1	M_2	M_3			
P_1	25] 4	6	8	50	(2)	25
P_2	13	11	10	35	(1)	
P_4	9	11	13	50	(2)	
	25	5	105			
	(5)	(5)	(2)			
	0					

	M_2	M_3			
P_1	5] 6	8	25	(2)	20
P_2	11	10	35	(1)	
P_4	11	13	50	(2)	
	5	105			
	(5)	(2)			
	0				

	M_3	
P_1	20] 8	20
P_2	35] 10	35
P_4	50] 13	50
	105	

		V_1	V_2	V_3	V_4	V_5	
		M_1	M_2	M_3	M_4	M_5	
$M_1 = 0$	P_1	25] 4	5] 6	20] 8 +	13	0	50
M_2	P_2	13	11	35] 10 −	~~20~~ 8 +	15] 0	70
M_3	P_3	14	30] 4	10	13	0	30
M_4	P_4	9	11	50] 13 −	20] 8	(0)	50
		25	35	105	20	15	

$C_{rs} = u_r + v_s$

$C_{11} = u_1 + v_1$

$4 = 0 + v_1 \Rightarrow v_1 = 4$

$v_2 = 6 \quad , v_3 = 8$

$C_{23} = u_2 + v_3$

$10 = u_2 + 8 \Rightarrow u_2 = 2$

$C_{24} = u_2 + v_4$

$8 = 2 + v_4 \Rightarrow v_4 = 6$

$C_{32} = u_3 + v_2$

$4 = u_3 + 6 \Rightarrow u_3 = -2$

$C_{43} = u_4 + v_3$

$13 = u_4 + 8 \Rightarrow u_4 = 5$

$C_{25} = u_2 + v_5$

$0 = 2 + v_5$

$v_5 = -2$

$u_1 = 0, \ u_2 = 2, \ u_3 = -2, \ u_4 = 5, \ v_1 = 4, \ v_2 = 6, \ v_3 = 8, \ v_4 = 6$
$v_5 = -2$

$d_{ij} = c_{ij} - (u_i + v_i)$

$d_{14} = c_{14} - (u_1 + v_4)$

$d_{14} = 13 - (0+6) = 7$

$d_{15} = 0 - u_1 + v_5 = 0 + 2 = 2$

$d_{21} = 13 - (2+4) = 7$

$d_{22} = 11 - 8 = 3$

$d_{31} = 14 - (-2+4) = 12$

$d_{33} = 10 - (-2+8) = 4$

$d_{34} = 13 - (-2+6) = 9$

$d_{35} = 0 - (-2-2) = 4$

$d_{41} = 9 - (5+4) = 0$

$d_{42} = 11 - (5+6) = 0$

$d_{44} = 8 - (4+6) = -2$

$d_{45} = 0 - (5-2) = \underline{-3}$

30 $50 - \theta$ $-\theta$

$5 + \theta$

$15 - \theta = 0$

$\theta = 15, 50$

$\underline{\theta = 15}$

		v_1 M_1	v_2 M_2	v_3 M_3	v_4 M_4	v_5 M_5	
$u_1 = 0$	P_1	25 | 4	5 | 6	20 | 8	13	0	50
$u_2 =$	P_2	13	11	10	8	0	70
$u_3 =$	P_3	14	30 | 4	10	13	0	30
$u_4 =$	P_4	9	11	15 | 13	20 | 8	15 | 0	50
		25	35	105	20	15	

$C_{15} = u_1 + v_5$

$C_{11} = 0 + v_1 \rightarrow 4 = v_1$

$v_2 = 6, \quad v_3 = 8$

$C_{23} = u_2 + v_3 \Rightarrow 10 = u_2 + 8 \Rightarrow u_2 = 2$

$C_{24} = u_2 + v_4$

$8 = u_2 + v_4 \Rightarrow \quad v_4 = 3$

$C_{32} = u_3 + v_2$

$4 = u_3 + 6 \Rightarrow u_3 = -2$

$C_{43} = u_4 + v_3$

$13 = u_4 + 8 \Rightarrow u_4 = 5$

$C_{45} = u_4 + v_5$

$0 = 5 + v_5 \Rightarrow v_5 = -5$

$d_{ij} = C_{ij} - (u_i + v_j)$

$d_{14} = 13 - (0 + 6) = 7$

$d_{15} = 0 - (0 - 5) = 5$

$d_{21} = 13 - (2 + 4) = 7$

$d_{22} = 11 - (2 + 6) = 3$

$d_{25} = 0 - (2 - 5) = 3$

$d_{31} = 14 - (-2 + 4) = 12$

$d_{33} = 10 - (-2 + 8) = 4$

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$d_{34} = 13 - (-2+3) = 12$

$d_{35} = 0 - (-2-5) = 7$

$d_{41} = 9 - (5+4) = 0$

$d_{42} = 11 - (5+2) = 4$

$d_{44} = 8 - (5+3) = 0$

$d_{45} = 0 - (5-5) = 0$

Cost $= 25\times4 + 5\times6 + 20\times8 + 70\times10 + 15\times13 + 20\times8 + 15\times0 + 30\times4$

$= 100 + 30 + 160 + 700 + 195 + 160 + 120$

Degeneracy and transportation problem:—

	X	Y	Z		
A	8	7	3	60	(4)
B	50/ 3	8	9	70	(5)
C	11	3	5	80	(2)
	50	80	80		
	(5)	(4)	(2)		
	0				

	Y	Z		
A	7	3	60	(4)
B	8	9	20	(1)
~~C~~	80/ 3	5	80	(2) 0
	80	80		
	(4)	(2)		
	0			

	Z	
A	60/ 3	60
B	20/ 9	20
	80	

		v_1	v_2	v_3	
		X	Y	Z	
u_1	A	8	7	60/ 3	60
$u_2 = 0$	B	50/ 3	8	20/ 9	70
u_3	C	80/ 11	3	5	80
		50	80	80	

$v_1 = 3, \quad v_3 = 9$

$C_{13} =$ ~~[illegible]~~

$3 = u_1 + 9 \Rightarrow u_1 = -6$ $(m+n-1)$

$C_{32} = u_3 + v_2$

$8 =$ ~~[illegible]~~ $u_3 + v_2$

		v_1 X	v_2 Y	Z $v_3 = 0$	
u_1	A	8	7	60 | 3	60
u_2	B	50 | 3	8	20 | 9	70
u_3	C	11	80 | 3	0 | 5	80
		50	80	80	

$u_1 = 3, \quad u_2 = 9, \quad u_3 = 5$

$C_{21} = u_2 + v_1$

$3 = 9 + v_1$

$v_1 = -6$

$C_{32} = u_3 + v_2$

$3 = 5 + v_2$

$v_2 = -2$ $, v_3 = 0$

$d_{ij} = C_{ij} - (u_i + v_j)$

$d_{11} = 8 - (3 - 6) = 11$

$d_{12} = 7 - (3 - 2) = 6$

$d_{22} = 8 - (9 - 2) = 1$

$d_{31} = 11 - (5 - 6) = 12$

$d_{ij} > 0$ all positive

$50 \times 3 + 80 \times 3 + 60 \times 3 + 20 \times 9$

$150 + 240 + 180 + 180 = 750$

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Assignment :—

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$x_{ij} = \begin{cases} 1 & \text{if } i^{th} \text{ person is assign to } j^{th} \text{ job} \\ 0 & \text{if } i^{th} \text{ person is not assign to } j^{th} \text{ job} \end{cases}$$

Q.

T/S	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

A-

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

row viz deletion

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	~~0~~
D	9	12	14	$\boxed{0}$

column viz deletion

I → A
II → C
III → B
IV → D

8 + 19 + 4 + 10 = 41 Ans

Q.

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

	I	II	III	IV	V
A	3	9	0	8	12
B	3	1	6	0	9
C	0	4	3	0	4
D	4	7	0	11	9
E	4	0	2	1	5

	I	II	III	IV	V	
A	2	9	$\boxed{0}$	8	8	✓
B	2	1	6	$\boxed{0}$	5	
C	$\boxed{0}$	4	3	✗	✗	
D	3	7	✗	11	5	✓
E	3	$\boxed{0}$	12	1	1	
			✓			

I → C
II → E
III → A
IV → B

Single line = No change
no line = subtract
cross line = addition

~~Min time = 137~~

	I	II	III	IV	V
A	$\boxed{0}$	7	~~0~~	6	6
B	2	1	8	$\boxed{0}$	5
C	~~0~~	4	5	~~0~~	$\boxed{0}$
D	1	5	$\boxed{0}$	9	3
E	3	$\boxed{0}$	4	1	1

I → A
IV → B
V → C
III → D
II → E

Min time = 11 + 6 + 16 + 17 + 10 = 60

	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

	a	b	c	d	e
A	30	0	45	60	70
B	15	0	10	40	55
C	30	0	45	60	75
D	0	0	30	30	60
E	20	0	35	45	70

	a	b	c	d	e	
A	30	$\boxed{0}$	35	30	15	✓
B	15	~~0~~	$\boxed{0}$	10	~~0~~	
C	30	~~0~~	35	30	20	✓
D	$\boxed{0}$	~~0~~	20	~~0~~	5	
E	20	~~0~~	25	15	15	✓
		✓				

	a	b	c	d	e
A	15	$\cancel{0}$	20	15	$\boxed{0}$
B	15	15	$\boxed{0}$	10	$\cancel{0}$
C	15	$\boxed{0}$	20	15	5
D	$\boxed{0}$	15	20	$\cancel{0}$	5
E	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

$e \to A$
$d \to B$
$b \to C$
$a \to D$
$d \to E$

Min time = 200 + 130 + 110 + 50 + 80
260 + 310 = 570

Ay

Unbalance Assignment Problem:—

Q. 5 jobs are assigned to 4 machine

job	A	B	C	D	E
I	9	7	6	2	0
II	6	6	7	6	0
III	5	3	4	4	0
IV	4	2	5	9	0
V	2	8	3	9	0

Ay—

job	A	B	C	D	E	
I	7	5	3	$\boxed{0}$	~~0~~	
II	4	4	4	4	$\boxed{0}$	✓
III	3	1	1	2	~~0~~	✓
IV	2	$\boxed{0}$	2	7	~~0~~	
V	$\boxed{0}$	6	~~0~~	7	~~0~~	
					✓	

subtract from 0 in row then subtract from min. in column

job	A	B	C	D	E
I	7	5	3	$\boxed{0}$	1
II	3	3	3	3	$\boxed{0}$
III	2	~~0~~	$\boxed{0}$	1	~~0~~
IV	2	$\boxed{0}$	2	7	1
V	$\boxed{0}$	6	~~0~~	7	1

I → D
II → E
III → C
IV → B
V → A

cost = 2 + 2 + 4 + 0 + 2 = 10 Ay

Travelling salesman problem:—

Q. A salesman must travel from cities to cities to maintain his account. He has to leave his home city A and visit another cities once & return home. The cost of going from one city to another is shown in the table find the least cost root.

	A	B	C	D	E
A	∞	4	10	14	2
B	12	∞	6	10	4
C	16	14	∞	8	14
D	24	8	12	∞	10
E	2	6	4	16	∞

	A	B	C	D	E
A	∞	2	8	12	0
B	8	∞	2	6	0
C	8	6	∞	0	6
D	16	0	4	∞	2
E	0	4	2	14	∞

	A	B	C	D	E
A	∞	2	6	12	$\boxed{0}$
B	8	∞	$\boxed{0}$	6	~~0~~
C	8	6	∞	$\boxed{0}$	6
D	16	$\boxed{0}$	2	∞	2
E	$\boxed{0}$	4	~~2~~	14	∞

A→E→A

B→C→D→B

Ambiguity

	A	B	C	D	E
A	∞	$\boxed{2}$	6	12	~~0~~
B	8	∞	$\boxed{0}$	6	~~0~~
C	8	6	∞	$\boxed{0}$	6
D	16	~~0~~	2	∞	2
E	$\boxed{0}$	4	~~2~~	14	∞

The solution cann't be found from this

Now

	A	B	C	D	E
A	∞	$\boxed{2}$	6	12	~~0~~
B	8	∞	$\boxed{0}$	6	~~0~~
C	8	6	∞	$\boxed{0}$	6
D	16	~~0~~	2	∞	$\boxed{2}$
E	$\boxed{0}$	4	~~2~~	14	∞

A→B→C→D→E

cost = 4 + 6 + 8 + 10 + 2

= 30